Mach, Einstein and Relationalism

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Thesis about Ernst Mach, General Relativity and Relationalism.
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November 4, 2009

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Abstract

A physical theory with only relations and no independent space and time is called a relational theory. Ernst Mach was a relationalist, but he did not have a full relational theory. Albert Einstein, inspired by Mach, sought for a complete relational theory. This search resulted in the general theory of relativity. Is this theory Machian? What does it mean for a theory to be Machian?

In this thesis I will argue that general relativity is relational, but that it is not Machian. This will be done by analysing Mach's critique to Newton's *Principia*. Mach introduced an alternative to Newton's law of inertia, in which he did not use the concept of absolute space. How should we interpret this alternative?

Also, Einstein's search for a relational theory of gravitation, and specifically properties which look non-Machian, will be studied. Finally, the relationalist interpretation of Carlo Rovelli will be sketched. This interpretation illustrates that general relativity suggests relationalism. However, the theory is not a pure description of experience, because it contains unobservables. In that respect, general relativity is not fully Machian.

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Introduction

Ernst Mach was a relationalist, but he did not have a full relational theory. Relationalism follows from the denial of the independent existence of space and time. If you want to do physics without the terms 'space' and 'time', you must talk about relations between objects. A physical theory with only relations and no space and time is called a relational theory.

Roughly, the opposite of relationalism is absolutism. For the latter term several different definitions are used in the literature. Most of these definitions are imprecise. If you realise that many times relationalism is defined as the opposite of absolutism, it is clear that relationalism does not have a precise consensus definition either.

In modern discussions the relationalists are exactly the ones opposing substantivalism, a type of absolutism. Substantivalists believe in the existence of space or space-time as some kind of *substance* or *container*. The modern relationalist is someone who denies this existence. Whether this implies Machianism is ambiguous. This depends on the definition of 'Machianism'.

The aim of this thesis is to answer the following questions: Are Mach and the modern relationalist the same? Do they even talk about the same thing? What is *Machianism* and its relation with relationalism? Is our best classical theory of space-time, the general theory of relativity, a Machian theory?

The first chapter tries to give an answer to the question what Mach talked about. What did he write about relationalism and what did he intend to accomplish with it? Is 'relationalism' the same as 'Machian'? What is a good definition of 'Machian'? In the second chapter Einstein's version of Mach's principle is introduced and we look at how Mach influenced the creation of the general theory of relativity. Also some foundational-historical issues, which are important in the light of Mach, of the theory is given. The third chapter will discuss the modern aspects of relationalism. We try to find the connections between Mach's words and modern potential relationalist theories and interpretations.

Chapter 1

Ernst Mach

Ernst Mach performed an experimental as well as theoretical research in the field of supersonic velocity. The Mach number is the ratio between the speed of a projectile and the speed of sound. Thus when this number is one, the speed of the projectile is the speed of sound, when it is two, the speed is two times the speed of sound et cetera. That is how the common people know of Mach. That is not what this chapter is about. This chapter is about his contributions to mechanics and cosmology, as well as related philosophical considerations. Specifically an important idea, which we now refer to as *Mach's principle*, will be discussed.

The literature of philosophy and history of Mach and his principle is confusing, because of different definitions and interpretations of the principle and simply because Mach did not invent the term 'Mach's Principle'. It is unclear whether Mach himself had such a principle. As we shall see, it is still a useful term.

The first aim of this chapter is to find out what a good definition of Mach's principle is. We want such a definition to be useful for the development of relational theories, as well as in line with Mach's intentions. To achieve this it is necessary to look at the primary sources, specifically what Mach wrote. It is a good idea to look at some secondary sources as well. In 1993 a conference was held in Tübingen about Mach's principle, and we will make extensive use of the proceedings of this conference [BP95].

Another aim of this chapter is to give an overview of what several writers in the proceedings of that conference think of the status of Mach's principle. Is it a new physical law, inconsistent with Newton's laws, or is it Newton's first law in a different form?

This chapter is organised as follows. First we will look at Newton's *Principia* and Mach's reaction to it. An analysis of Newton's work and Mach's response is made to find out about Mach's intentions. For this we will primarily use Mach's *Mechanik* [Mac60]. After that the texts of Barbour [BP95, pp. 6-8, 214-231], Norton [BP95, pp. 7-57] and Borzeszkowski and Wahsner [BP95, pp. 58-64] from the proceedings will be discussed. In the succeeding chapter Einstein's definition of Mach's principle and how he used it is discussed.

1.1 What Mach wrote

Mach's *The Development of the Principles of Dynamics*[1] was first published in 1883, after which a number of new editions and translations have been published. For this chapter the 1960 English translation is used.[2] The second chapter of the book covers Newtonian mechanics, in a historical and philosophical way. After two sections about Galileo and Huygens, he explains Newtonian mechanics. In the sixth section Mach quotes extensively from Newton's *Principia*. As we will see, Mach disagreed with Newton's views on space, time and motion.

1.1.1 Newton's *Principia*

In "The Mathematical Principles of Natural Philosophy" (in short *Principia*, [New87]) Newton set out the laws of motion. Preceding these laws of motion there is a section called 'Definitions'. The first part of that section contains several definitions of (what we now call) mass, momentum, inertia, centripetal force et cetera. After the definitions follows a *scholium* in which he expounds on his ideas about space, time and motion. However, these are terms, he notes, that need no definition. The scholium is a philosophical clarification.

After the *scholium* Newton set out the laws of motion, the axioms as he called them. Mach did not have problems with the axioms, which he explains in sections three and four of the second chapter of his *Mechanik* [Mac60, pp. 226-264]. The disagreement was about the preceding *scholium*, in which he described metaphysical entities like space and time. Mach attacked Newton's metaphysics, not his physics. We define *metaphysical* as everything that is not part of a certain theory which tends to describe reality. Mach means with 'metaphysical' everything that falls outside experience.[3]

In the following paragraphs we will evaluate Mach's reaction to Newton's explanation of the terms *time* (paragraph I of the *scholium*) and *space* (paragraph II). Newton saw time as absolute, as a flow, independent of anything else. As we will see, Mach's reaction was that we cannot measure time, there is no absolute time. Space, according to Newton, is absolute, unchangeable, independent of anything else. Mach's reaction was that absolute space and absolute motion do not exist, there are only relative distances and relative motion.

Newton wrote the following about *time*:

---

[1] *Die Mechanik in Ihrer Entwicklung Historisch-Kritisch Dargestellt*

[2] The 1893 translation was approbated by Mach: "I can testify that the publishers have supplied an excellent, accurate, and faithful rendering of it" [Mac60, p. xxv] and we can only assume that our use of the 1960 edition is fine as well. We must be careful, though. E.g. Barbour found some comments which appeared only in older editions, which might be relevant for the interpretation of Mach's work (see note 2 on [BP95, p. 230]).

[3] This seems compatible with above definition, because for Mach a theory should only describe experience. This is not trivial, because theories tend to describe much more than only experience. See section 3.2.

> "Absolute, true and mathematical time, of itself and by its own nature, flows uniformly on, without regard to anything external. [...] But the flow of *absolute* time cannot be changed. Duration, or the persistent existence of things, is always the same, whether motion be swift or slow or null" (Newton as quoted in [Mac60, p. 272]).

Mach's response was:

> "It would appear as though Newton in the remarks here cited still stood under the influence of the mediæval philosophy, as though he had grown unfaithful to his resolves to investigate only actual facts. When we say a thing A changes with the time, we mean simply that the conditions that determine a thing A depend on the conditions that determine another thing B." [Mac60, p. 272]

This is the opposite of Newton's view, in which time is a uniform flow. Mach meant to say that absolute time does not exist and that the apparent phenomenon of time is because of relative changes. A clock is not an indicator of an absolute time and passage thereof, but it is a change measured relative to other things that change, e.g. the rotation of the earth or the duration of a period of the transition between the two hyperfine levels of a specific atom. In Mach's own words: "[Time] is an idle metaphysical conception." [Mac60, p. 273].

About *space* and *motion* Newton wrote:

> "Absolute space, in its own nature and without regard to anything external, always remains similar and immovable. [...] Absolute motion is the translation of a body from one absolute place to another absolute place" (Newton as quoted in [Mac60, pp. 276, 277]).

He acknowledged that "we use, in common affairs, instead of *absolute* places and motions, *relative* ones; [...] The effects by which absolute and relative motions are distinguished from one another, are centrifugal forces, or those forces in circular motion which produces a tendency of recession from the axis." Newton illustrated this with his famous thought experiment of the bucket.

Bucket experiment

> "If a bucket, suspended by a long cord, is so often turned about that finally the cord is strongly twisted, then is filled with water, and held at rest together with the water; and afterwards by the action of a second force, it is suddenly set whirling about the contrary way, and continues, while the cord is untwisting itself, for some time in this motion; the surface of the water will at first be level, just as it was before the vessel began to move; but, subsequently, the vessel, by gradually communicating its motion onto the water, will make it

> begin sensibly to rotate and the water will recede little by little from the middle and rise up at the sides of the vessel, its surface assuming a concave form. (This experiment I have made myself.) [...] At first, when the *relative* motion of the water in the vessel was *greatest*, that motion produced no tendency whatever of recession from the axis; the water made no endeavour to move towards the circumference, by rising at the sides of the vessel, but remained level, and for that reason its *true* circular motion had not yet begun. But afterwards, when the relative motion of the water had decreased, the rising of the water at the sides of the vessel indicated an endeavour to recede from the axis; and this endeavour revealed the real motion of the water" (Newton as quoted in [Mac60, pp. 277–278]).

Three stages in this experiment can be distinguished and we may add a fourth for completeness:

0. Bucket still, water still $\Rightarrow$ water flat

1. Bucket rotating, water still $\Rightarrow$ water flat

2. Bucket rotating, water rotating $\Rightarrow$ water concave

3. Bucket still, water rotating $\Rightarrow$ water concave

Stage 0 is the situation just after the cord is turned about and filled with water. Stage 1 is when the bucket "is suddenly set whirling about the contrary way." Interestingly the relative speed between the bucket and the water is maximised, while the water stays flat. This might be explained by the introduction of absolute space. The water is at rest with respect to absolute space and therefore it will not rise up at the sides of the bucket. At stage 2 the water does rise up at the sides, but its speed with respect to the bucket is decreased. This can be explained by absolute space as well. For completeness sake, a final stage 3 might be added, where the bucket stops rotating but the water still rotates. As to be expected, the water is still concave.

All stages of the experiment might be explained by introducing absolute space. Our experiences from the experiment do not make sense if we only look at the relative speed between the bucket and the water. Stage 1 and 2 of the experiment are contradictory with that idea. Mach did not want to introduce an absolute space, so how did Mach interpret Newton's bucket experiment? Instead of invoking absolute space, he brought up the idea that the form of the water's surface can be explained in relative terms if we take into account the earth and distant stars. The relative speed of the water with respect to all massive objects in the universe might explain the form of its surface. Mach uses this idea, as shown in the proceeding subsection.

Next, Mach quoted Newton on the difficulty to distinguish true (absolute) from apparent (relative) motion. Even though the bucket experiment seemed to make at least clear that these types of motion are (in some cases) distinguishable, he presented yet another thought experiment (the globes), but this does not add anything which is essential to the argument.[4] Mach's reaction:

> "If, in a material spatial system, there are masses with different velocities, which can enter into mutual relations with one another, these masses present to us forces. We can only decide how great these forces are when we know the velocities to which those masses are to be brought. *Resting* masses too are forces if *all* the masses do not rest. Think, for example, of Newton's rotating bucket in which the water is not yet rotating. If the mass m has the velocity v_1 and it is to be brought to the velocity v_2, the force which is to be spent on it is $p = m(v_1 - v_2)/t$, or the work which is to be expended is $ps = m(v_1^2 - v_2^2)$.[[5]] *All* masses and *all* velocities, and consequently *all* forces, are relative. There is no decision about relative and absolute which we can possibly meet, to which we are forced, or from which we can obtain any intellectual or other advantage." [Mac60, p. 279].

Forces are expressed in velocity differences and time differences, both of which are relative by definition. Therefore force can always be put in relative terms as well. Acceleration (by for instance rotation) is not a way to make this distinction. According to Mach, it is impossible to make a distinction between absolute and relative motion.

This is no proof that there are no absolute spatial properties, but it does show that these are not empirically accessible. It seems that for Mach empirically inaccessible is the same as not existing, or at least it is not meaningful to talk about empirically inaccessible things. In subsection 1.2.3 is more about Mach's philosophy and how this relates to thought experiments like Newton's bucket, and counterfactuals.

Mach finished this section of *Die Mechanik* with the following remark:

> "When quite modern authors let themselves be led astray by the Newtonian arguments which are derived from the bucket of water, to distinguish between relative and absolute motion, they do not reflect that the system of the world is only given once to us, and the Ptolemaic or Copernican view is our interpretation, but both are equally actual. Try to fix Newton's bucket and rotate the heaven of fixed stars and then prove the absence of centrifugal forces." [Mac60, p. 279]

[4] Some authors say it does. Cf. [Ear89, pp. 62-63] and [Ray87, pp. 7-9].

[5] Here t is a time difference which is sufficiently small for the first formula to hold within the framework of Newtonian mechanics.

One cannot actually do this experiment, fix Newton's bucket and rotate the heaven. It is a counterfactual — what we should concern ourselves about, is the actual only world. The fact that Mach pressed that the world is only given once will be interesting for the discussion of the positions of Norton and Barbour (section 1.2).

1.1.2 Mach's alternative

The largest part of section II.VI, "Newton's Views of Time, Space, and Motion" [Mac60, pp. 271-297] is an attack on the views of Newton. In this subsection we will only discuss the part where Mach introduced his own alternative.

Mach introduced a system of coordinates for a body K, relative to (and fully determined by) bodies A, B, C et cetera. This is not a coordinate system situated in absolute space, but a relative coordinate system. If we now introduce a second body K' close to K and move them both with constant velocity with respect to the other bodies the following formula would be obtained [Mac60, p. 286]:

$$\frac{d^2r}{dt^2} = \frac{1}{r}\left[a^2 - \left(\frac{dr}{dt}\right)^2\right] \ . \tag{1.1}$$

Here r is the distance between K and K', a is a constant, dependent on the directions and velocities of the two bodies, and t is the time. Now if a is equal to dr/dt, equation 1.1 reduces to zero acceleration.

Mach's next step is to consider these two bodies independent of each other, but each still relative to the distant bodies A, B, C et cetera. We come to the following formula, which expresses that the mean acceleration of K with respect to the other bodies is zero [Mac60, p. 287]:

$$\frac{d^2}{dt^2}\left(\frac{\Sigma mr}{\Sigma m}\right) = 0 \ . \tag{1.2}$$

The summation in the numerator is over the masses m and the respective distances r. This means that the mass is weighed with the distance, thus distant masses are more significant in this equation than nearby masses. The denominator is the total mass in the universe and functions as a normalisation factor. The equation represents a law of inertia, which is the phenomenon that a force is needed to change the speed or direction of a massive object.[6] Just like Newton's first law this establishes inertial frames of reference thus defining inertia.

Mach notes the approximate equivalence between equation 1.1 and 1.2: "The latter expression is equivalent to the former, as soon as we take into consideration a sufficient number of sufficiently distant and sufficiently large masses" [Mac60,

[6] The larger the mass of an object, the larger the force is needed to change its state of motion. For instance, changing the state of motion of a heavy cannon ball from rest to say 50 km/h is more difficult than doing the same to a marble.

p. 287]. Two pages further Mach describes what has been done: "We have attempted in the foregoing to give the law of inertia a different expression from that in ordinary use. This expression will, so long as a sufficient number of bodies are apparently fixed in space, accomplish the same as the ordinary one." (ibid, p. 289). These two quotes imply that Mach's law of inertia (equation 1.2) is not equivalent to Newton's first law, but it might be approximately equivalent.

When one thing is *approximately equivalent* to another, the difference between the two might be negligible, but even then it is not an equivalence. Does this mean that Mach opted for a new physical law? In the next section this question and several views on what Mach actually intended will be discussed.

1.2 Mach's Intentions

1.2.1 Mach's Principle

In the introduction to this chapter we ran into the term "Mach's principle", without defining it in a precise way. That job is not trivial and the reason for doing this now is to present it without bias, with the exception of the above treated primary sources. As we have seen Mach did not use the word *principle.* Einstein, as well as contemporaries, used this term and probably the following is meant:

> **Mach's Principle.** *Inertial frames must be defined with respect to other bodies and not (absolute) space and time.*

This means that *inertia* is induced by other masses (and not space). This is a useful definition of Mach's Principle (two capitals). Also, this is in line with Einstein's formulation of Mach's principle. Einstein said that for general relativity to be consistent, inertia cannot be defined with respect to space, but it must be defined with respect to other masses. This will be treated in subsection 2.1.2.

The principle is just a specification, not yet an implementation. Mach's formula (1.2) is an implementation of the principle. When we talk about Mach's Principle, we mean this specification or proposal, and we cannot trivially deduce any law of physics (an implementation) from it. We can only check whether an implementation obeys the specification.

Newton's first law expresses that the acceleration of a body *with respect to absolute space* is zero (when no force is impressed). This is incompatible with Mach's Principle. There is not really a principle which implements Newton's first law, but his law is based on the assumption that there exists independent, absolute space and time.

Mach's Principle is a reaction to Newton's views on space and time. Mach assumed that space and time are idle metaphysical concepts, and thus the only way to define inertia is with respect to other bodies. When we look for an implementation of Mach's Principle, we must think of a replacement of Newton's first law, because this is the law of inertia in Newtonian mechanics. Intuitively Mach's formula (1.2) does precisely that: it expresses that the mean acceleration of a body with respect to other bodies is zero if and only if no external forces work on the body.

Before we think that it is obvious what Mach intended with his formula as well as the principle, we must confuse ourselves by looking at what other writers had to say about this. Our first victim will be John Norton who argues that it is not clear whether Mach intended to introduce a new physical law (subsection 1.2.2). However, he is straightforward about formula 1.2 – this is just a rewritten version of Newton's first law. Then Julian Barbour's point of view is discussed (subsection 1.2.4). He does not agree with Norton and showed that this is a genuine new physical law and that only therefore Mach opted for a new theory, one which is incompatible with Newtonian mechanics. A philosophical intermezzo about counterfactuals is in between (subsection 1.2.3). Finally the short article of Horst-Heino v. Borzeszkowski and Renate Whasner is discussed (subsection 1.2.5). In the article they explicitly state that Mach searched for a reformulation of Newtonian mechanics, not a new theory.

1.2.2 Unclear intentions

Norton wrote: "The core idea is that the inertial forces acting on an accelerating body arise from an interaction between that body and other bodies. The idea is not so much a proposal of a definite, new physical law; rather it is the prescription that such a law should be found. The law recommended is only loosely circumscribed" [BP95, p. 9]. This prescription is what Norton calls Mach's principle[7]; the law itself may be, for instance, Mach's formula (1.2). Even though he considers this 'not so much' a proposal of a new physical law, he leaves this option open.

From Mach's *Mechanik* [Mac60] Norton notes two themes which are important for Mach [BP95, p. 12].

- Science must aspire to provide simple and economical descriptions of experience.
- Newton's absolute space, time and motion are metaphysical idle notions.

Both themes reflect Mach's positivist philosophy (see subsection 1.2.3). Mach's response to Newton's bucket experiment (see the paragraph *Bucket experiment* in the previous section) follows.

[7]It is not equivalent with our definition, because it talks about inertial forces, which is more than a definition for inertia. An implementation of the principle as defined by Norton may also supersede Newton's second law (the law of resultant force). This difference in the definition of Mach's principle will turn out not to be very important, thus we may use Norton's Mach's principle and our Mach's Principle interchangeably.

> "Newton's experiment with the rotating vessel of water simply informs us, that the relative rotation of the water with respect to the sides of the vessel produces *no* noticeable centrifugal forces, but that such forces *are* produced by its relative rotation with respect to the mass of the earth and the other celestial bodies. No one is competent to say how the experiment would turn out if the sides of the vessel increased in thickness and mass till they were ultimately several leagues thick. The one experiment only lies before us, and our business is, to bring it into accord with the other facts known to us, and not with the arbitrary fictions of our imagination." [Mac60, p. 284]

According to Norton one can read this in two ways. The first is that we need a new physical mechanism for inertia. This is what we mean by 'Mach's Principle'; it would replace Newton's first law. The second, a more subtle reading: it recalls the two themes mentioned above and it is unclear whether a new physical mechanism is needed. Then it is still possible that Mach wished to rewrite Newton's laws in such a way that the term 'space' does not occur in that description, but qua formalism it should still be equivalent to Newton's laws. Norton calls these readings incompatible, but within the second reading the possibility is still open that we may need a new physical mechanism: a law which would not be compatible with Newton's first law and would replace it.

In the proceeding section Norton makes clear that there are exactly two possibilities [BP95, pp. 15 ff.]:

- Mach intended a mere redescription of mechanics.
- Mach intended to introduce a new physical mechanism.

According to Norton there are two ways of reconciliation. One is that Mach didn't want to commit himself to a new physical mechanism; he was unwilling to accept what his own system produced. The other is that "a causal connection for Mach is merely a functional dependence extracted from experience" [BP95, p. 28]. This is much different from the common view of causation. So if Mach talks about the causal relation between a body with the rest of the universe, it is not obvious at all whether this implies a new physical law.

In subsection 1.2.1 we introduced the words 'law' and 'principle'. The relation was roughly: 'a law is an implementation of a principle'. We didn't use the word *mechanism*. If we would think about causation, we normally think about a mechanism, something we can picture. Mach does not think like this. For him a law is a universal regularity. This hints to the reason why Mach did not speak of a 'principle': there is no law to implement, only a universal regularity. Such an empirical regularity is just something derived from our senses. To test an empirical regularity with respect to a beforehand introduced principle might give a contradiction. Obviously the principle must be removed or adjusted, so why introduce such a principle in the first place?

For the argument's sake, say Mach did intend to introduce a new physical mechanism. Newton's inverse square law is certainly not a causal relation in the usual sense.[8] It was a functional relation. Just as Newton did not have a causal, constructive explanation of this law, Mach did not have one for his new physical mechanism. In a way, just like Newton didn't 'explain', Mach didn't 'explain'. This does not have any consequences for whether Mach tried to introduce a new physical mechanism. Therefore, Norton's second argument, the functional dependence argument, is not valid.

Reconciliation

Norton preferred his second way of reconciling the two views on Mach's principle – a causal connection is just a functional dependence extracted from experience. This is different from the common view of causation, Norton explained. Newton's inverse square law is understood as causation. The sun causes an acceleration of the earth. This should also be the case in an otherwise empty universe, but Mach "does not allow, in general, this assuming away of the other masses of the universe" [BP95, p. 28]. The functional relation between the sun and the earth is defined only in our actual universe. This is the only experience we can have about this relation.

The same argument, Norton argued, can be applied to the bucket experiment. According to Mach one cannot say anything about a situation where there exists only the bucket and the water. The functional relation between the water and the bucket is only in the actual universe, where there is also the earth and other celestial bodies. This means that Mach never had to propose a new physical mechanism. Any significant change in the celestial bodies (like removing all of them) would be relevant for such a new physical mechanism, but there is only one universe, and in that universe such a change does not occur. It is not sensible to talk about an almost empty universe and therefore a new physical mechanism is not needed.

Norton's only objection to this functional relation extracted from experience stems from correspondence between Mach and Einstein about inertia. Einstein wrote to Mach about inertia using counterfactual systems. Mach allowed Einstein to attribute such things to Mach's work, without objections.[9] This is unexpected, because for Mach counterfactual systems do not have any meaning.

[8] Newton's inverse square law is often criticised because it is only a functional relation and not something with a constructive picture. The mechanism is not clear. When we define 'causal explanation' by 'functional dependence' this problem trivially vanishes.

[9] There exists a written objection supposedly by the hand of Ernst Mach to Einstein's theory of relativity. This is the preface to *Die Prinzipien der physikalischen Optik* [Mac21], which was published after Mach's death by Mach's son Ludwig. However, it is not clear whether Mach actually wrote this. It may have been a forgery by his son Ludwig. See for this statement [Wol87] and a critical notice to this work [DiS90].

1.2.3 Counterfactuals and reality

Philosophical introduction

Normally in physics we talk about theories as if they describe reality or as if they are at least empirically adequate. A theory can be applied to a large set of models. These models depend on boundary conditions or initial conditions. Sometimes we can create different sets of boundary conditions, so that we can test whether different models satisfy a certain theory. When we have tested a lot of models with success, the theory is true or at least empirically adequate. This means, by induction, that counterfactual models would follow the same laws as the tested models. We could do more experiments and, without any surprise, find out that these other models also satisfy the theory.

We cannot do this for cosmological models, because there is only one universe. That is, for the purpose of testing a theory with cosmological models, we cannot apply induction. When we talk about the whole universe, we talk about one model only. There is no sense in imagining, say, an almost empty universe with only a rotating bucket with water. However, for some cosmological theories a cosmological model is just an extrapolation of smaller scale models. For instance, classical mechanics is such a theory which works on smaller scale (e.g. the solar system), but is universal. That means that it works anywhere in the universe, for any conceivable submodel of the universe. The extrapolation is the application of the theory to the universe. We are then left with only one model which is actually testable. Any other model would be a counterfactual, one which is principally not testable.

The question is whether it is all right to do the described extrapolation or whether it is not, that is whether we should apply a theory to exactly one cosmological model. Physicists before and especially after Mach would argue for the former. Mach did not think it is alright to talk about counterfactuals; the universe is given only once. These two views are incompatible with each other.[10]

Mach's philosophy

Mach wrote the following about actual facts:

> "It is scarcely necessary to remark that in the reflections here presented Newton has again acted contrary to his expressed intention

[10]Even though most physicists say it is okay to talk about counterfactuals, Norton thinks that Mach's position is acceptable. In September 2008 in an e-mail to me he wrote: "It is a thorough-going positivism that, in my view, does not fit well with developments in physics that followed Mach (and a lot that went before him)." He argued that Mach's views were rather common for a lot of physicists for a long time. For Mach, physics is directly connected with experience and has no place for metaphysics. Most physicists nowadays are not positivists and modern physics is often not directly connected with experience, Norton reasoned. Especially therefore we should not analyse Mach from a modern perspective.

> only to investigate *actual facts*. No one is competent to predicate things about absolute space and absolute motion; they are pure things of thought, pure mental constructs, that cannot be produced in experience. All our principles of mechanics are, as we have shown in detail, experimental knowledge concerning the relative positions and motions of bodies. Even in the provinces in which they are now recognised as valid, they could not, and were not, admitted without previously being subjected to experimental tests. No one is warranted in extending these principles beyond the boundaries of experience. In fact, such an extension is meaningless, as no one possesses the requisite knowledge to make use of it." [Mac60, p. 280]

From this it is clear that Mach wished for a description of the actual world and nothing else. Anything that is not empirically accessible is meaningless. This includes imaginary universes, space and time.

1.2.4 New physical mechanism

A way to explain Mach's silence about Einstein's misattributions (page 18) is that they were no misattributions. Mach possibly did look for a new physical mechanism. This is the point of view which Julian Barbour defends. At the Tübingen conference he argued that Mach's principle has to be a new physical mechanism and also that general relativity is a perfectly Machian theory [BP95, pp. 214–236].

There are several arguments Barbour made for Mach advocating a new physical mechanism. The primary argument is that no nontrivial mechanical system can possibly agree with Mach's formula (1.2) as well as Newton's first law. Another argument is that Mach praised the work of Hofmann, who unambiguously talked about a new theory, different from Newton's. Hofmann is clear in his writings that he intended an actual replacement of Newtonian mechanics, one which is (empirically) incompatible with Newton's laws. Several other arguments are made by Barbour, but these two seem the most straightforward and convincing.

Formula 1.2 might be compatible with Newtonian mechanics for specific models at specific points in time. Mach would certainly have expected this law to hold over time. However, over a time span any nontrivial dynamical system cannot be described by Mach's formula as well as Newton's law of inertia (the first law). We will illustrate this with the following example. Consider a model with a lot of masses, which is at least at some point in time equivalently described by Newton's first law as well as Mach's formula. Now consider a body somewhere at the border of the universe[11] moving for instance one degree East with respect to the rest of the universe. The resulting time-evoluted model cannot be described

[11] It might be as well in the outer parts of a finite mass distribution in an infinite universe. For the argument it suffices to assume that the body is far enough and not of negligible mass.

precisely by Newton's first law and at the same time by equation 1.2, because this equation is heavily dependent on distant masses, while the effect of distant masses converges to zero in all of Newton's laws. Specifically Newton's first law is not even dependent on any mass except for the test body itself. Newtonian mechanics is simply incompatible with Mach's formula, because Mach's formula contains the masses of the bodies of the universe and Newton's first law does not.

The other argument is written in an endnote of Barbour's article. Mach referred to Hofmann (Barbour's italics): "I have in front of me also a lively, clear text written in a very popular style by W. Hofmann [. . .] who seems unaware of the controversy and who *seeks the solution in almost the same ways as I did*" [BP95, p. 230, note 2]. In his article (part of it in [BP95, pp. 128–133]) Hofmann talks about experimental results, not about compatibility with Newtonian mechanics. Statements like "complete theory of inertia" (p. 129), "the overwhelming influence is surely to be ascribed to the infinitely great mass of the heaven of the fixed stars" (p. 132), "[it would] be very interesting to consider the Foucault pendulum experiment [...] in order to establish experimentally the influence exerted by the earth, sun, and the other masses on the pendulum" (ibid) and "a means to establish experimentally the extent to which these inertial influences depend on the magnitude of the masses [...]" (ibid), show that Hofmann searched for (and formulated) a new physical theory. As we have seen above, Mach said that Hofmann searched for a solution in almost the same way as he did. This indeed underlines Barbour's point of view, that Mach advocated a new physical mechanism.

Doubt

If we now want to have a serious discussion about Borzeszkowski and Wahsner's argument that Mach did not want a new physical mechanism, but just tried to introduce a rewritten version of Newtonian mechanics, we must create some doubt surrounding Barbour's rather straightforward arguments.

First consider the first argument. Mach's formula 1.2 indeed describes a new physical mechanism, because the dynamics it describes is incompatible with Newton's first law. However, this *should* have been a vector equation and also the particle's own mass *should* have been included. These important details might be ignored, but because of these errors we must consider the possibility that we should not conclude anything from Mach's equation. If Mach was wrong about these details, then it may have been a mistake as well that he included all those distant masses in the formula! Moreover, if he was wrong about these simple mathematical details, why would he not have made any mistakes in the calculations where he checked whether there are any models for which Newton's law gives the same results as Mach's equation? Possibly the answer is that he did not check this. Therefore, the fact that his equation is not equivalent with Newtonian mechanics, is not a good argument that Mach's equation is supposed to be a new physical mechanism.

Besides considering the mathematics and the physics, Mach added a lot of philosophy and interpretation to his formula (1.2). In two pages after his formula Mach wrote: "It is impossible to say whether the new expression would still represent the true condition of things if the stars were to perform rapid movements among one another." [Mac60, p. 289]. Then it seems that formula 1.2 is no universal formula, or at least not in a modern interpretation of 'universal'. With this I mean that a 'universal law' should be valid for any model of our theory. This includes counterfactuals. For Mach there is only one universe and the very idea to "perform a rapid movement" of distant bodies is nonsense: "[w]e must, on the contrary, *wait* until such an experience presents itself" (ibid). This seems to refute the experiment with the distant body moving one degree East (above mentioned illustration of the first argument for a new physical mechanism). It does not happen. It is not the case that we tried to consider counterfactual models of the universe, but only the actual one. For our nontrivial universe (the actual universe) Mach's formula can simply not be compatible with Newton's laws of motion.

To create doubt for the second argument I would say Mach either did not read Hofmann very well or in the word *'almost'* he tried to encode the 'detail' that he does not agree with the fact that Hofmann's approach is to look for a new physical theory. However, it is doubtful for Mach to speak so high of Hofmann when he does not agree with him on this issue.

1.2.5 Reformulation

In "Mach's Criticism of Newton and Einstein's Reading of Mach: The Stimulating Role of Two Misunderstandings" [BP95, pp. 58–66] Borzeszkowski and Wahsner contended that Mach tried to give a reformulation of Newtonian mechanics. The basis for this is firstly that the removal of metaphysical elements does not actually change the theory, and secondly, that Mach didn't understand a part of Newtonian mechanics, namely the concept of inertial frames.

Here follows the first argument:

> "As far as mechanics is concerned, Mach considered it correct but, for historical reasons, represented by Newton in a manner containing a lot of metaphysical elements. Therefore, he intended to remove these elements by reformulating mechanics, and the result was his critical-historical account of mechanics" [BP95, p. 59].

A theory usually consist of physical as well as metaphysical terms. The metaphysical terms may safely be removed from the theory. From this point of view we must conclude that it is possible to give a reformulation of classical mechanics, without metaphysical terms like space and time.[12] This suggests that Mach intended just that. However, from the fact that Newton's theory contains a lot of metaphysical elements, does not follow that Mach intended to remove these elements *by reformulating mechanics*. Possibly Mach thought it would be necessary to adjust Newton's laws, in order to throw out the metaphysical ballast. It is not clear whether Mach allowed that or really pressed a reformulation.

The second argument: Mach misunderstood mechanics, especially the space in which Newtonian mechanics is defined. Mach was under the impression "that the space of Newtonian mechanics is a rigid background given once and for all" [BP95, p. 60]. In modern formulations of classical mechanics, space is not absolute in this sense. Space is the totality of all inertial systems. Furthermore, Borzeszkowski and Wahsner noticed that "in Mach's day classical mechanics was taught in a version which indeed was loaded with metaphysical ballast" [BP95, p. 60]. Absolute space (in a full-Newtonian space-time sense, see appendix A) is such metaphysical ballast. To work without this metaphysical ballast, Lange's operational definition of 'inertial frame' might be used, as set out in the following subsubsection.

Inertial systems

In 1886 the mathematical physicist Ludwig Lange clarified the notion *inertial system*, by giving an operational definition. This definitions will be expounded here.

Start with a mass-point A (on which no force is exerted) in arbitrary motion. Introduce a coordinate system in which A moves in a straight line. Introduce a second mass-point B and a third point C, which are also in arbitrary motion (and not subjected to external forces). It is still possible to define a coordinate system in which all three points move in straight lines. When we introduce a fourth point, this is not trivial. The first three mass-points *define an inertial system*. When we introduce a fourth point which is not subjected to any forces, it will also follow a straight line in the given coordinate system, but this is not by definition. It is Newton's first law, or rather the operational definition of it. Note that the straightness of the first three particles is a matter of convention or definition; the straightness of a fourth particle (or more particles) is no convention, but an expression of Newton's first law.

Lange simply gave an operational definition of the inertial system and used this to formulate Newton's first law. Metaphysical terms like space and time are not used in his formulation. He only talks about mass-points. This seems very much in line with Mach's goal: a description of mechanics (specifically inertia) without the use of metaphysics. It also seems like a rewritten version of Newton's first law, thus no new physical mechanism is introduced.

[12]For a proof of concept, cf. Lynden-Bell's relative Newtonian mechanics [BP95, pp. 172–178]. This theory is equivalent with Newtonian mechanics, in all cases when the angular momentum of the whole universe is zero.

Borzeszkowski and Wahsner argue that Mach did not know how to handle Lange's method: "[He had] problems accepting Lange's definition of inertial frame without changing his own criticism of Newtonian mechanics. [Mach's] way out of this dilemma is to say that Lange's answer to the question as to the reference system of mechanics and thus to the notions of space is purely mathematical, while his own is physical." [BP95, p. 61].[13]

Concluding remarks

So, where are we now? Norton has doubts about whether Mach meant to introduce a new physical mechanism or whether he just meant to restate Newton's laws in a relational way. He prefers the latter and this has to do with Mach's "extremely restrictive view of causation". Borzeszkowski and Wahsner are sure about what Mach meant: he wanted to give a reformulation of Newtonian mechanics. However, Barbour is also sure and said that Mach did wish to introduce a new physical mechanism.

The whole discussion about whether Mach meant to introduce a new physical mechanism is possibly somewhat void. I think that Mach himself did not have a concrete answer to the question whether a new physical mechanism was necessary or desirable. He just wanted a relational theory, in which 'space' did not have any independent meaning. He was not sure what this implied for the physics and did not polarise to one point of view. This is clear throughout most of Mach's quotes in the article of Norton.

In chapter 3 the context created in this chapter will be used to find an answer to our question: Do Mach and the modern relationalist talk about the same thing and are their intentions the same? Even though in this chapter no definite answer is given to what Mach actually intended, a central equation (1.2) is interpreted and we expounded on Mach's philosophy. Now we can compare this with modern theories and conceptions.

[13] Lange's construct forms neo-Newtonian space-time, a term that John Earman introduced in his book "World Enough and Space-Time" [Ear89]. This is not the same as the space-time that Earman calls 'Machian'. See appendix A for an overview of the different classical space-times. On the basis of Earman's definitions one might be persuaded to agree with Mach's point of view that Lange's definition of inertial frame does not fulfil the task of removing space and time as independent objects.

Chapter 2

Albert Einstein

Einstein searched for a theory of relativity for arbitrary motion, which means that the laws of physics are not only equivalent in different inertial frames of reference, but as well in non-inertial frames. Einstein was motivated by this search, because of ideas of Ernst Mach. Mach believed that space and time are not absolute, that is, they are not independent of physical objects. Therefore the laws of motion should be invariant in different frames of reference. This brought Einstein to think that it should not matter for the laws of nature whether you are in an inertial frame of reference or an accelerating one.[1] Demanding this is troublesome, because the experiences of an observer in accelerating frames are different from the ones in inertial frames. In the end Einstein did find a general theory of relativity, but whether this theory is equivalent for all frames of reference is unclear.

This chapter is divided in three sections. The first will discuss the ideas Einstein used from Mach and his interpretation of Mach. The most explicit idea that Einstein used is what he called 'Mach's principle'. The term 'Mach's Principle' has already been discussed in the previous chapter, but not in a historically accurate way. The original term 'Mach's principle' is introduced by Albert Einstein. The second section expands on the general theory of relativity. A short historical overview is given and some core ideas of general relativity, which are very much related to Machian points of view, are explained. The third section discusses vacuum solutions of the Einstein field equations. These solutions imply independence of the metric from matter, thus suggest that the general theory of relativity is not Machian.

[1] The thought experiment with the elevator as will be discussed in subsection 2.2.3 is the main motivation for this idea.

2.1 Inspiration from Mach

In the previous chapter we looked at Mach's philosophy. In this section, we will see more clearly in what way Mach's philosophy influenced Einstein.

2.1.1 Mach according to Einstein

From Abraham Pais's famous biography of Einstein [Pai83] we learn that three themes reflect Mach's influence on Einstein.

1. Relativity of *all* motion;
2. Scientific methodology;
3. Origins of inertia.

The first theme is taken by Einstein to show that the laws of nature are the same, independent of the state of motion of the frame of reference. It will be treated in subsection 2.2.1.

The second is about Mach's philosophy of the method of science. For Mach *sensations* were nature's real elements. Sensations or experiences alone are not enough to do science. We need to make abstractions, or ideas associated to experience in the most economical way. He only defined what science should do, not what nature actually is.[2]

Einstein was strongly influenced by this theme, at least in his early years. This does not mean he was Machian in any precise sense, but that he was inspired by Mach. Over time he took theoretical abstractions more seriously and thus used a methodology which was far from Machian. Even though Einstein became more realistically inclined, also in his later years he still had very empiristic, very Machian moments.[3]

From the third theme Einstein formulated Mach's principle. As will be shown, this definition of Mach's principle is similar to the definition given in the previous chapter.

2.1.2 Mach's principle

Mach's principle advocates that inertia is a consequence of the matter distribution of the universe. The principle will be defined more precisely further on. 'Inertia' is a quantity which expresses a body's resistance to change its velocity. Einstein wanted the metric field $g_{\mu\nu}$[4] to determine inertia, while $g_{\mu\nu}$ is determined by the

[2] Mach expounds on the subject of an economical description of experience in [Mac60, pp. 577–595].

[3] For an essay on Einstein's change of scientific methodology, see [Hol73, pp. 219–259].

[4] This is the metric tensor, a tensor field which defines the geometry of space-time.

mass distribution of the universe. Thus inertia would be fully determined by the mass distribution. This turned out to be wrong in general relativity, because De Sitter showed that there are non-trivial solutions of the Einstein equations to an empty universe.

In Einstein's 1917 article "Cosmological Considerations in the General Theory of Relativity" [SKK96, pp. 541-551] his ideas about inertia are formulated. This is a step towards his formulation of Mach's principle. Einstein wrote:

> "My recently conceived opinion about boundary conditions in infinite spaces brought me to the following considerations. In a consistent theory of relativity there can be no inertia with respect to space, but merely inertia relative between masses. When I move a mass sufficiently far away from all other masses of the world, its inertia should disappear." [5]

In other words, according to Einstein, *inertia* is something induced by other masses (and not space). It follows from his ideas about boundary conditions of the universe; this will be discussed in subsection 2.3.2.

In an article of 1918, "On the Foundations of the General Theory of Relativity", Einstein formulated Mach's principle:

> "*Mach's principle*: The G-field [i.e. the Einstein tensor] is fully determined by the masses of bodies. Mass and energy are equal, as a result of the special theory of relativity, and the energy is formally described by the symmetrical energy tensor ($T_{\mu\nu}$). This implies that the G-field is conditioned and determined by the energy tensor of matter." [6]

The G-field, the Einstein tensor, is defined in terms of the metric tensor, the Ricci tensor and the Ricci scalar: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$. It is proportional to an expression of the energy-momentum tensor, or the matter distribution. The resulting equations are the familiar Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} ,$$

where G is Newton's constant of gravity.

[5] "Meine bis vor kurzem gehegte Meinung über die im räumlich Unendlichen zu setzenden Grenzbedingungen fußte auf folgenden Überlegungen. In einer konsequenten Relativitätstheorie kann es keine Trägheit gegenüber dem *Raume* geben, sondern nur eine Trägheit der Massen gegeneinander. Wenn ich daher eine Masse von allen anderen Massen der Welt räumlich genügend entferne, so muß ihre Trägheit zu Null herabsinken." [SKK96, p. 544].

[6] "*Machsches Prinzip:* Das G-Feld ist *restlos* durch die Massen der Körper bestimmt. Da Masse und Energie nach den Ergebnissen der speziellen Relativitätstheorie das Gleiche sind und die Energie formal durch den symmetrischen Energietensor ($T_{\mu\nu}$) beschriebenwird, so besagt dies, daß das G-Feld durch den Energietensor der Materie bedingt und bestimmt sei." [SJI$^+$02, pp. 38–39]

We may speak of inertia, if the G-field is interpreted as a measure of inertia. At this point it is not clear whether the G-field can be interpreted in this way. For this reason the general theory of relativity should first be studied more closely.

2.2 Theory of General Relativity

The discovery of the theory of general relativity was not easy, and it took a long time. Extensive literature exists on the history of the development of the theory. In this section important subjects for the foundations of the general theory of relativity are discussed. These are the general principle of relativity and general covariance.

The first subsection is about the possibility of a generalisation of the special principle of relativity, leading to a fully relativised theory. It will be suggested that general covariance is the tool to fulfil this requirement. The second subsection is a historical overview of the development of the general theory of relativity. In this overview it will be shown that general covariance is not a formal requirement for the generalisation. Kretschmann even argued that general covariance is physically void. In the last subsection the *equivalence principle* is treated, which is more essential than general covariance for general relativity. Finally, it will be shown, however, that general covariance can be used in a physically meaningful way, when it is related to the equivalence principle.

The subsequent section will expand on the subject of De Sitter solutions. Willem de Sitter showed that the Einstein field equations yield non-trivial solutions[7] of a universe without matter. However, Einstein's opinion was, in line with his reading of Ernst Mach, that the $g_{\mu\nu}$ field should be fully determined by matter and not be able to exist without it.

2.2.1 Generalisation of the Principle of Relativity

On the one hand, the laws of physics are the same in all inertial frames. This is true in Newtonian mechanics, as well as special relativity.[8] On the other hand, the laws are different from frames which are not inertial. Einstein tried to formulate a theory which was fully relativised. This means that the laws of physics are the same in any frame of reference, inertial or not. This is a generalisation of the special principle of relativity. The tool he used to fulfil this relativity requirement was general covariance. General covariance states that one can do arbitrary coordinate transformations to any part of space-time such that the resulting situation is physically equivalent to the original. In his "The

[7] *Non-trivial* means anything that is different from Minkowski, that is, solutions with non-zero curvature.

[8] The name of this theory is chosen the same as the principle.

Foundation of the General Theory of Relativity" of 1916 [SKK96, pp. 284–338] Einstein defined general covariance as follows:

> "The universal laws of nature are to be expressed by equations, which hold good for all coordinate systems, i.e. are covariant with respect to arbitrary substitutions (generally covariant)." [9]

Even though he introduced the term *general covariance* in 1912, this meaning will be used throughout the rest of the discussion about general covariance.

2.2.2 Historical overview

In 1912 Albert Einstein introduced *general covariance* with the goal of generalising the special principle of relativity to include, besides inertial frames, accelerated frames. Such a formulation would be a relevant ingredient for a relativised theory of motion and, as Einstein noted in the beginning of 1913, the problem of gravity would also be solved with generally covariant equations.

In 1913 the *Entwurf* theory appeared in a paper by Einstein and his friend Marcel Grossmann [KSKR95, pp. 303–339]. It was an outline of a generalised theory of relativity and a theory of gravity. Even though the theory had many characteristics in common with the general theory of relativity (which was published two years later), the field equations in this paper were not generally covariant. In this important respect the theory was imperfect.

At the end of 1913 Einstein said that this was acceptable. Einstein made two arguments against general covariance. One of these arguments is his hole argument. The other is a consideration of conservation laws, which will not be discussed here, cf. [KSKR95, Doc. 13, p. 318]. The hole argument proceeds as follows. Consider a space-time with some configuration of matter, which defines a metric field. Define a 'hole' as a connected part of space-time. Now smoothly transform the coordinates in the hole, but leave the space-time coordinates outside the hole unchanged, such that the transition from the inside of the hole to the outside the hole is smooth as well. The field values will be different also. This is called a passive transformation. Now we may replace the transformed coordinates for the original coordinates, because it is only a *label.* The resulting transformation gives a different field on the same coordinates, or equivalently, the space-time points get moved around, with their values of the metric field. This means that all matter outside the hole does not determine what will happen inside the hole. Therefore, general covariance is a bad idea.[10]

Einstein attempted to find, based on the *Entwurf* theory, valid field equations. He did not succeed. This is why Einstein reintroduced general covariance in the

[9] "Die allgemeinen Naturgesetze sind durch Gleichungen auszudrücken, die für alle Koordinatensysteme gelten, d. h. die beliebigen Substitutionen gegenüber kovariant (allgemein kovariant) sind." [SKK96, p. 291]

[10] A more precise treatment of the hole argument is given in subsection 3.1.2.

end of 1915, which resulted in his general theory of relativity. Einstein stated that the differences between the fields which are described in the hole argument are 'formal' differences, not 'physical'.

In 1916 Einstein introduced a new argument in favour of general covariance: the *point coincidence argument*. This is the argument that coincidences of physical objects alone, for instance the crossing of the world lines of two particles, are relevant for the description of the world. Other space-time points do not tell anything that is observable, and thus these should be removed from the formalism all together. The coincidences are preserved under smooth transformations and thus smooth transformations on a specific model do not yield physically different models. This is precisely what happens if general covariance is assumed. In a generally covariant theory smooth transformations can be applied on a model and interpreted in an active sense, like in the hole argument, without the physical content of that model changing. This idea suggests that general covariance is the right approach.[11]

In 1917 Kretschmann criticised Einstein's coincidences solution to the hole argument. Kretschmann argued that general covariance is physically void, because one can make *any* physical theory generally covariant. One may even formulate Newtonian mechanics in a generally covariant form, but it would yield a complicated theory.[12] Einstein agreed with Kretschmann, but at the same time he kept the idea alive that general covariance is also something physical and that it is important in theory choice. The usage of general covariance in Newtonian mechanics is without regard to the actual goal of general covariance. Einstein used general covariance in a way to give a Machian, relational description. That description is something physical; it is not a mathematical statement.[13]

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[11] The set of transformations which conserves point coincidences between particle is much larger than the set which *smoothly* transforms the metric field. In subsection 3.1.3 we will find a modern formulation of the point coincidence argument, where also the $g_{\mu\nu}$ field will be something that can be coincided with particles. Einstein merely tried to make an argument for general covariance. For this it was enough to mention that at least some observables are conserved under smooth transformations. He did not intend to give a full characterisation of what is observable.

[12] See for instance Donald Lynden-Bell's relative Lagrangian formulation of Newtonian mechanics [BP95, pp. 172–178], in which he rewrites the kinetic energy relative to an axis moving with velocity $\mathbf{v}(t)$. Then he minimises the kinetic energy over all possible choices of $\mathbf{v}(t)$ and uses this, in combination with the already relative potential, to write down the new relativised Lagrangian.

[13] In later comments about general covariance Einstein said it was for a large part a mathematical statement. He remarked sometimes that general covariance is just a mathematical characteristic of general relativity (or other theories), while other times he emphasised the physical reasons. The following events might be a reason for the slight change in attitude. In 1915, before Einstein's publication of the general theory of relativity, the German mathematician Erich Kretschmann introduced the point-coincidence argument (but not by its name). Einstein used this idea without giving him credit. This might be the main motivation for Kretschmann to argue with Einstein concerning the point-coincidence argument. In return Einstein did not

2.2.3 Equivalence principle

An essential ingredient for the general theory of relativity is the equivalence principle. First we will explain what the principle entails. After that we explain in what way it is a generalisation of the special principle of relativity.

Consider yourself standing in an elevator without windows in deep space, accelerating with 9.8 m/s^2. How would you know that you are in deep space, and not simply experiencing earthly gravitation, the elevator standing firmly on the earth? The answer is that you do not and that you cannot find out by means of mechanical experiments. Similarly we could compare an elevator in free fall with one floating freely (without rotation) in deep space. Again, we cannot make out the difference with a mechanical experiment. This is because all objects accelerate in the same way, independently of their mass, in a gravitational field. The fact that locally there is no difference between the acceleration of different objects in a gravitational field, is called the *weak equivalence principle*.

In his general theory Einstein defined an inertial frame of reference as one that includes frames in free gravitational fall.[14] The *strong equivalence principle* states that the laws of physics are the same in curved and flat space-times. This we see when the laws are written in geometric form (that is, not in components). Equivalently one can define the strong equivalence principles as: *it is possible to define an inertial frame of reference at any point in space-time*. An example of where the strong equivalence principle is applied follows.

Consider the $g_{\mu\nu}$ field of an elevator in free fall, described in coordinates x^λ. We can now do a coordinate transformation of this field $g_{\mu\nu}(x^\lambda)$ to a different field in different coordinates, $g'_{\mu\nu}(x'^\lambda)$. This is equivalent to the elevator in vacuum, without acceleration. We can replace the x' by x again and our resulting actively transformed field is $g'_{\mu\nu}(x^\lambda)$. Here we used general covariance to illustrate that an elevator in free fall is equivalent to an elevator in inertial motion in vacuum.

2.3 Solutions to the Einstein field equations

As will be shown in the following paragraphs, for Einstein there was a problem with his field equations, as published in 1915. Solutions to the Einstein field equations for empty universes are possible, even in a finite closed universe, as De Sitter showed. These models have a metric, independent of any matter, because there is no matter. Einstein argued against these models, because of this

let Kretschmann trample over his arguments which were important for general covariance from a physical point of view. Therefore Einstein did not agree with the physical implications of Kretschmann's counterargument, even though Einstein did agree with his argumentation from a formal point of view.

[14]This has to be local, i.e. local enough in comparison with the divergence of the gravitational field. The modern definition of inertial frame of reference may be 'following a geodesic'. This formulation suggests that gravity is a geometric feature, instead of an external force.

independence. In line with Mach, he believed that the properties of space-time should be fully determined by matter. At some point Einstein admitted that there are non-trivial solutions of an empty universe, subsequently he let go of the idea that the metric should be fully determined by matter.

2.3.1 Boundary conditions

In November 1916 the Dutch physicist and astronomer Willem de Sitter wrote Einstein about his thoughts on the relativity of inertia [SKJI98a, Doc. 272]. De Sitter raised the issue of the demand that the metric approximates, for large distances, the Minkowski metric. Such a condition introduces an absolute element in the theory. It is not matter which determines the Minkowski metric at large distances, but it is an independent boundary condition imposed on the Einstein field equations.

Einstein agreed with De Sitter that in general relativity, as it stood, boundary conditions (together with matter) are necessary, to fully determine the metric. Einstein tried to eliminate this absolute element by postulating degenerate values for the metric field at infinity. This means that a continuum of different values of the field are valid solutions for a certain mass distribution, that is, these values are not determined a priori. It is a demand which replaces the demand of the trivial boundary condition (namely, the Minkowski metric for large distances). In this way only the mass distribution determines the metric and the inertial mass of test particles at infinity disappear. De Sitter showed that this is problematic, because a degenerate $g_{\mu\nu}$ field would not be generally covariant. Moreover, distant masses outside the visible part of the universe would be necessary. Because of these objections, Einstein put aside his idea.

In his cosmology article of February 1917 [SKK96, Doc. 273, pp. 540–552] Einstein resolved this by only considering finite universes without boundaries. To accomplish this for a globally stable universe, he introduced a constant λ (the *cosmological constant*): "[...] we can multiply $g_{\mu\nu}$ on the left side of the field equation with an unknown universal constant λ, without the removal of general covariance."[15] The new system of equations then is:

$$G_{\mu\nu} - \lambda g_{\mu\nu} = -x(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T),$$

where $T_{\mu\nu}$ is the energy-momentum tensor and T is its trace. These field equations are still generally covariant. This is an important demand, because then physical

[15] "Das Gleichungssystem ([13]) erlaubt jedoch eine naheliegende, mit dem Relativitätspostulat vereinbare Erweiterung, welche der durch Gleichung (2) gegebenen Erweiterung der Poissonschen Gleichung vollkommen analog ist. Wir Können nämlich auf der linken Seite der Feldgleichung (13) den mit einder vorläufig unbekannten universellen Konstante $-\lambda$ multiplizierten Fundamentaltensor $g_{\mu\nu}$ hinzufügen, ohne daßdadurch die allgemeine Kovarianz zerstört wird" [SKK96, Doc. 43, p. 550].

laws are independent of the choice of coordinates. Without the knowledge of the expansion of the universe, the demand for a globally static universe was sensible for Einstein. After the discovery of the expansion of the universe, the cosmological constant was discarded by Einstein.

2.3.2 De Sitter solution

In March 1917 De Sitter gave a non-trivial solution to these modified Einstein field equations without matter. De Sitter started his letter to Einstein: "I have found that the equations $G_{\mu\nu} - \lambda g_{\mu\nu} = 0$, which are your equations (13a) *without matter*, are satisfied by $g_{\mu\nu}$ [...]",[16] after which he put forward his solution for $g_{\mu\nu}$.

The existence of non-trivial solutions of the universe without matter implies that matter is not necessary to determine the metric. That is, one could put some test particles in this empty universe, and they would move completely according to the metric of De Sitter.[17] The paths of these particles would not be trivial as in Minkowskian space-time. They can be complex, but fully determined by the metric, because there is no matter. Einstein thought this was unsatisfactory, because this means that the metric can exist independently of the mass distribution.

Problem 0: The metric can exist independently of matter.

This problem is of a philosophical type and not hard physics. Even though we will see more physical arguments against the Sitter solutions, problem 0 is the origin of it.

Physical problems

Shortly after De Sitter's letter, he wrote him [SKJI98a, Doc. 317] that his solutions were also *physically* unsatisfactory, because they contained singularities.

Problem 1: The metric is singular.

Concerning this objection, Einstein wrote: "So, it seems to me, that your solution does not correspond with any physical possibility. $g_{\mu\nu}$ and $g^{\mu\nu}$ must be (together with your previous derivations) continuous everywhere."[18]

Then Einstein repeated the main argument (problem 0), which was the actual reason for the whole debate: "According to me it would be unsatisfactory if a world without matter were possible. The $g^{\mu\nu}$ field should be *fully determined*

[16] "Ich habe gefunden, dass man den Gleichungen $G_{\mu\nu} - \lambda g_{\mu\nu} = 0$ also Ihre Gleichungen (13a) *ohne Materie*, genügen kann durch die $g_{\mu\nu}$ [...]" [SKJI98a, Doc. 313, p. 414].

[17] A test particle is a mass which has no significant influence on other particles or the metric.

[18] "Es scheint mir deshalb, dass Ihrer Lösung keine physikalische Möglichkeit entspricht. Die $g_{\mu\nu}$ und $g^{\mu\nu}$ müssen überall (samt ihren ersten Ableitungen) stetig sein" [SKJI98a, Doc. 317, p. 422].

by matter and not be able to exist without it".[19] He continued highlighting his Machian views:

> "This is in essence what I understand what the requirement of relativity of inertia means. One can just as well speak about the 'material contingency of the geometry'. As long as this requirement was not fulfilled, for me the objective ['Ziel'] of general relativity was not yet entirely reached. This was first produced by λ."[20]

Einstein was not satisfied with the possibility of a non-trivial empty spacetime. The 'objective' or 'designation' was not reached, according to Einstein, because a non-trivial empty universe should not be a possible solution. It is in contradiction with his own Mach's principle. Even though we cannot say he (still) wished for a fully relative theory of motion, there are Machian ideas playing a role in his reasoning.

Response from De Sitter

From a sanatorium in Doorn (for treatment of tuberculosis) De Sitter wrote in April 1917 a reply to Einstein, concerning *world matter*, the singularity objection (problem 1) and the homogeneity argument. [SKJI98a, Doc. 321]

De Sitter introduced the term 'world matter', matter added to ordinary matter with the purpose of explaining inertia. This extraordinary matter was needed for explaining the boundary conditions that Einstein introduced. De Sitter pointed out that these boundary conditions appeared to be degenerate and therefore a bad idea.

About the singularity objection of Einstein De Sitter wrote: "I think that [the singularity] is just apparent, and therefore I imagined my, truly hyperbolic, world as a spherical one."[21] The singularity that Einstein found, due to the stereographic projection of a hyper-hyperboloid, is in temporal infinity. It is not a physical singularity, but can be removed by a coordinate transformation.[22] In the light of later insights in general relativity and differential geometry we know

[19] "Es wäre nach meiner Meinung unbefriedigend, wenn es eine denkbare Welt ohne Materie gäbe. Das $g^{\mu\nu}$-Feld soll vielmehr *durch die Materie bedingt sein, ohne dieselbe nicht bestehen können*" [SKJI98a, Doc. 317, p. 422].

[20] "Das ist der Kern dessen, was ich unter der Forderung von der Relativität der Trägheit verstehe. Man kann auch ebensogut von der 'materiellen Bedingtheit der Geometrie' sprechen. Solange diese Forderung nicht erfüllt war, war für mich das Ziel der allgemeinen Relativität noch nicht ganz erreicht. Dies wurde durch das λ-Glied erst herbeigeführt" (ibid).

[21] "Ich denke mir aber dass sie nur scheinbar ist, und daher rührt dass ich meine, wirklich hyperbolische, Welt als eine Spherische dargestellt habe." [SKJI98a, Doc. 321, p.427-8]

[22] Another, more common, example of a coordinate (i.e. not physical) singularity in general relativity is the one at the event horizon of the Schwartzschild solution. We can transform the coordinates to Eddington-Finkelstein coordinates where the metric behaves perfectly nice at the event horizon (cf. [Car97, pp. 183-4]).

that Einstein had not identified physical singularities, but that these only were an artifact of the choice of coordinates.

In [SKJI98a, Doc. 351] two other issues with the De Sitter solutions were raised:

Problem 2: The universe is not static.

Problem 3: The universe has a preferred centre.

About the first De Sitter asked Einstein why the universe *would* be static. The Milky Way, for instance, is not.

> "We only have an instantaneous picture of the world, and from the fact that we do not see any big changes, we are only allowed to conclude that we do not see large changes on the photograph, and we cannot conclude that this will always be the case."[23] Besides, De Sitter argues in the same letter, even on a large scale we cannot assume that matter is homogeneously distributed. Actually, "all our observations speak against this."[24]

Therefore, even if we say that our local part of our universe is static, it might not be in the non-visible part of the universe.

Acceptation

In a July 1917 letter to De Sitter Einstein wrote that De Sitter's solution had a preferred centre (problem 3) [SKJI98a, Doc. 351]. This was, like the singularities, just an artifact of the chosen coordinates [SKJI98a, Doc. 356, note 8]. In June 1918 Einstein had an exchange of letters with Felix Klein. Klein wrote him in detail why the singularities were just artifacts of the chosen coordinates. Einstein replied: "You are absolutely right. The De Sitter world is in principle free from singularities [...]"[25] His singularity argument, as well as the preferred centres argument, was erroneous.

Einstein's argument against De Sitter's solution concerning the stability of the universe (problem 2) was a wrong bet as well, for now is known that the universe is not static.

[23] "Wir haben von der Welt nur eine Momentphotographie, und wir können und dürfen daraus, dass wir auf der Photographie keine grosse Veränderungen sehen, *nicht* schliessen dass alles immer so bleiben wird als in dem Momente wo die Aufnahme gemacht worden ist." [SKJI98a, Doc. 321, p. 428]

[24] "alle unsere Beobachtungen sprechen dagegen" (ibid).

[25] "Sie haben vollkommen Recht. Die De Sitter'sche Welt ist an sich singularitätsfrei und ihre Weltpunkte sind alle gleichwertig" [SKJI98b, Doc. 567, p. 809].

Concluding remarks

To conclude, Einstein came up with several problems with De Sitter's solution, mainly to maintain his position concerning a Machian description of the world. These problems were misconceptions, which were resolved over time. This led to Einstein's acceptance of De Sitter's solution. This is compatible with the idea that Einstein left Machianism and became more of a realist. Furthermore, in the next chapter will be shown that general relativity can be interpreted in a partly Machian way.

Even though Einstein did not succeed in accomplishing what he originally wanted, he did revise several of his views, under which the Machian empiricism. Later he was more of a scientific realist, not so much about space-time but about the metric field. Therefore, all of this is not so pessimistic as it may look, it was just, as written in the subtitle of Michel Janssen's 2005 article, a "bumpy road to general relativity" [Jan05].

Chapter 3

General Relativity as Partly Machian

In the previous chapter Albert Einstein's search for a general theory of relativity was expounded. In this search an important lead was Mach's Principle, a notion defined in chapter 1. Recall that it means that inertial frames must be defined with respect to other bodies and not absolute space and time. The nature of chapter 2 was mostly historical. Several important foundational aspects, like the De Sitter solutions, the equivalence principle and general covariance were discussed as well. One important philosophical question surrounding the subject of general relativity and Ernst Mach was taken up in historical context and was not yet discussed in a foundational manner. This question is whether general relativity is truly Machian. It will be discussed in this chapter.

Recall that the term *Machian theory* was defined as complying with two themes, identified by John Norton (section 1.2.2):

- Absolute space, time and motion are metaphysical idle notions.
- Science must aspire to provide economical descriptions of experience.

In the first section of this chapter different types of absoluteness in space-time theories are introduced and compared to general relativistic space-time. It is argued that space-time in general relativity is not absolute in any sense and a relational interpretation of the general theory of relativity is introduced. According to that interpretation, absolute space, time and motion are indeed metaphysical idle notions. Recall from chapter 1 that 'metaphysical' means that it is not part of physics. For Mach this is the same as that it is not part of experience.

Mach was a strong positivist and Einstein was influenced by Mach's philosophy. In spite of this, it will be shown in the second section that general relativity is not a pure description of experience[1] and in this sense it is not a Machian theory. One might say that the general theory of relativity is only partly a Machian theory.

3.1 Absolute space

In this section it will be argued that if general relativity is interpreted as a relational theory, the solution to the hole argument becomes trivial and absolute space and time are, in Mach's words, metaphysical, idle notions. [Mac60, p. 273]

In the first subsection different types of absoluteness will be compared with each other. It is claimed by the philosopher Michael Friedman that we should differentiate between these types. This will be done and my conclusion will be that space-time of the general theory of relativity is not absolute in any sense.

The second subsection expands on the hole argument (see section 3.1.2 for its historical context), of which the discussion was an important episode in the development of general relativity and possesses a relevance that extends to the philosophy of space and time in general.

The last subsection expounds on Carlo Rovelli's interpretation of general relativity. (Carlo Rovelli is a philosophically inclined physicist, specialised in canonical quantum gravity.) His interpretation washes away the dilemma of the hole argument as well as most types of absoluteness. Rovelli's arguments are not unique in this respect, but in my opinion they clarify the relationalist interpretation of general relativity.

3.1.1 Types of absoluteness

Friedman contrasts between three different kinds of absoluteness [Fri83, pp. 62–64], namely:

Absolute–Relational Does space-time exist independent of matter? Space-time of a theory is absolute when it does; the theory is relational when space and time do not exist independently of matter.

Absolute–Relative Is the choice of certain reference frames relevant for the interpretation of space-time? For instance in neo-Newtonian mechanics[2] is

[1] Implicitly it is then shown that the theory is not a *purely economical* description of experience. This is a logical escape from defining what is 'purely economical'. The trouble with defining such a term is that different incompatible definitions might be relevant. For instance, for Mach, throwing out metaphysical objects like absolute space and time is economical. For Einstein and many others simplicity is at least as relevant for a theory being economical. These ideas are not always compatible. A relational theory of Newtonian mechanics is economical in the sense that there is not absolute space and time, but known formulations the formalism is difficult (e.g. Lynden-Bell's article in [BP95, pp. 172–178]) and thus not economical.

[2] This is Newtonian mechanics in a space without an absolute rest frame. See appendix A.

no preferred inertial reference frame, which means that the space in this theory is not absolute but relative.[3]

Absolute–Dynamical Does matter influence the structure of space-time? It does not in Newtonian physics or special relativity, which means that space-time in these theories is absolute, but it does in general relativity, which means that space-time in general relativity is dynamic.

These different types of absoluteness are a priori logically unrelated. For instance, absolute in the first sense (non-relational) does not mean that matter cannot effect space-time. This means that the absolute–relational type of absoluteness is in principle unrelated to the absolute–dynamical distinction (cf. [Fri83, pp. 64–70]). To get a grasp of the different types of *absoluteness* and understand why these often get confused, I will give an example of space-time which has a strong absolutist character.

Consider full Newtonian space-time. This is 'absolute' in a very strong sense, the strongest sense that is considered here.[4] Here follow several properties of full Newtonian space-time:

> It is a global foliation[5] of the four-manifold, resulting in a stack of Euclidean submanifolds E^3. The manifold can thus be denoted as $E^3 \times \mathbb{R}$. Our type of absoluteness demands that space and time are independent of matter, and matter does not have an effect on space nor time. Furthermore, spatial points have an identity over time. This means that a point (x, y, z) on time t is by definition the same as the point (x, y, z) on time $t' \neq t$. Space is Euclidean and thus homogeneous and isotropic[6], and it does not change, which means it is non-dynamical.

[3]This can be seen as part of the structure of space, but Newton's ideas were different. Even though Newton incorporated Galilei invariance (that mechanical laws are empirically equivalent in different inertial frames), he held fast on an absolute frame of rest. The type of space-time resulting from this is called full Newtonian space-time, while incorporating inertial structure, as is usually done for Newtonian mechanics, results in neo-Newtonian space-time. See appendix A.

[4]If an absolute centre or preferred position would be added, this type of space-time is called Aristotelian space-time (confer [Ear89, ch. 2] and appendix A), for Aristotle assumed that the centre of the earth was the centre of the universe, and everything has a natural position, yielding an absolute coordinate frame. The argument in this paragraph works for full Newtonian space-time, as well as for Aristotelian space-time.

[5]A foliation is a local product structure of a manifold. 'Local' signifies that it does not have to be the whole manifold, but that the product structure only applies to local coordinate patches. With 'global foliation' I mean that there is a global product structure. In the case of Aristotelian space-time it is simply a Cartesian product between a Euclidean three-space E^3 ($\mathbb{R}^3$ would be a standard representation) and time $\mathbb{R}$.

[6]Homogeneous means that space looks locally the same at every point. This property follows from isotropy: uniformity in all directions. Euclidean space, a space complying with the five axioms of Euclid, is uniform in all directions (isotropic) and therefore looks locally the same

Often when we speak of 'absolute', we mean one or several of these properties. It might be useful to discriminate between these. In many sources this is done only implicitly, so that the reader cannot tell what definition of absoluteness is used. We will try to avoid that situation here, by using Friedman's different types of absoluteness.

None of the mentioned types of absoluteness are as strong as 'absolute' in the strongest sense. These three types of absoluteness follow from the strongest sense. This means that at least Newton and contemporaries did not bother to discriminate between the different types of absoluteness. It happened to take until Friedman's analysis before anyone made such a clear distinction. Friedman showed that when someone uses the word 'absolute' without an explanation, it is very ambiguous. For a clear discussion one should differentiate between the different types of absoluteness.

Absolutism in general relativity

What does all this mean for the general theory of relativity? Is space-time in general relativity absolute in some sense? Friedman's definitions will be considered, but first absoluteness in the strongest sense of absolute, full Newtonian space-time, will be shown to be absolute in all senses. This means that it is the wrong space-time for the general theory of relativity. Generally, according to general relativity, space (as a slice of space-time) is not Euclidean, so space-time certainly is not $E^3 \times \mathbb{R}$. It is dynamical as well, influenced by matter. We can be certain that in general relativity space-time is nothing like full Newtonian space-time.

Friedman's first definition is about the independence of the existence of space-time. In standard text books of general relativity space-time is built up by first defining a manifold (points with neighbourhoods) and putting extra structure on top of that (connection, curvature, ...). From this it may seem that space-time exists independently of matter. In the next subsubsection it will be shown however, that this type of absoluteness (space-time point substantivalism) is only apparent. Besides this type of independent existence of space-time, there are however non-trivial solutions to the Einstein field equations for empty universes (see the discussion between Einstein and De Sitter in subsection 2.3.2). In this sense one might say that space-time is absolute, as discussed in the next subsubsection.

General relativity is not absolute according to Friedman's second definition. In full Newtonian space-time (appendix A) is a privileged inertial frame, the rest frame. In Galilean space-time (or neo-Newtonian space) a part of the absolute character of full Newtonian space-time is thrown out of its structure, namely the rest frame. All inertial frames are ontologically equivalent in Galilean space-time.

everywhere (homogeneous). Generally there are many other manifolds which are isotropic and thus homogeneous. There are also many manifolds which are homogeneous and not isotropic and finally there are manifolds which are inhomogeneous and thus anisotropic.

In general relativity there are even larger equivalence classes of models,[7] which means that space-time is not absolute in the absolute–relative sense.

According to Friedman's third definition of absoluteness, general relativistic space-time is not absolute either. Gravity is described by a dynamical space-time or a dynamical field. Matter influences space-time: it is the cause of curvature.

Substantivalism in general relativity

As has been shown, space-time of general relativity is both relative and dynamical. One could still argue in favour of *substantivalism*, however. This is Friedman's first type of absoluteness, the opposite of *relationalism*. The central question in this is whether in a theory space and time exist independently while they contain physical objects, or the theory only describes spatio-temporal relations between physical objects.

In this sense a theory is absolute or substantivalist when space and time exist independently and contains physical objects, like a container. This is a statement about ontology. Not only particles and fields exist, but also a space-time manifold, that is, the whole set of space-time points is considered as existing independently of its contents. On the other hand, a theory is relational or not substantivalist when it describes merely the relations between physical objects. In this the ontology consists only of particles, fields and relations. There is no independent space-time manifold.

In the context of general relativity it is useful to distinguish the first type of absoluteness, namely substantivalism, in two subtypes. One is (space-time) point substantivalism, which states that every point of the manifold has a unique identity, ontologically different from each other. The other is substantivalism of space-time as a whole. This is a type of substantivalism which is weaker than point substantivalism and is relevant in the context of vacuum solutions of the Einstein Field Equations. These two types of substantivalism will now be discussed in the context of general relativity, starting with point substantivalism.

Subtype I: Point Substantivalism One difficulty with interpreting general relativity lies in its formalism. The formalism seems rather substantivalist, because space-time is defined as a four-dimensional manifold.[8] This means that

[7]Consider a model and apply a diffeomorphism. This gives an empirically equivalent model. The set of all such models which are diffeomorphic with respect to each other is called an equivalence class. Formally, if M is a model in the set X of valid space-time models according to general relativity, the equivalence class of element M in X is a subset of all elements in X which are empirically equivalent to M, i.e. $[M] = \{x \in X | x \sim_L M\}$ (where $\sim_L$ stands for Leibniz equivalence).

[8]A manifold is a collection of points with a closeness relation. To be precise, a four-manifold is a collection of points which can be labelled with four coordinates. Locally one can define for every point a neighbourhood which is homeomorphic with $\mathbb{R}^4$. For the formalism consult e.g. [Car97, pp. 31–54] or [Wal84, pp. 11–28].

there is a container, or substance, in which physical objects can be placed. When this is the case, the space-time of the theory has a point substantivalist nature. If a metric is introduced on the manifold, normally this is written in a system of coordinates. The metric is defined with respect to the manifold, that is, in the container.

However, general relativity might be interpreted in a different, non-substantivalist, way. This happens when we look for the intrinsic structure of the theory. This proceeds as follows.

The metric field can be smoothly transformed, by moving points. Such a transformed field is still a solution to the Einstein equations. Given one solution of a coordinate dependent metric, one can perform an enormous amount of smooth transformations, which result in empirically equivalent situations. As shall be shown in subsection 3.1.3, these solutions might be interpreted as equivalent, in an ontological sense. The resulting equivalence classes of metrics form the 'real' coordinate independent metric, or the geometry of space-time.

We end up with general relativity without coordinates. Naturally we still have the tendency to think of the field in some representation of coordinates. In general relativity there is no canonical or privileged representation, there is just the coordinateless field, the geometry.

The set of equivalence classes of metrics can be considered as the original set of metrics, 'divided' by the symmetry of the theory. In the case of a generally covariant theory, like general relativity, this is the general covariance group.[9] Effectively, this 'division' results in removing the manifold from the formalism. There might be some structure left, but in any case, this 'division' interpretation forces one to be no point substantivalist. Subsection 3.1.2 about the hole argument illustrates this extensively, but here we will already provide a sketch of the argumentation against the existence of a manifold.

One might perform an active point transformation in a smooth way, yielding an equivalent situation. There is no meaning in talking about space-time points as having a unique identity, different from each other. That is why we must do away with the manifold. The field might be *represented* by functions of field patches on $\mathbb{R}^4$, but the specific representation has nothing to do with reality. One must only consider the geometries of space-time. These geometries are coordinate independent. The fact that the theory's solutions are intrinsically independent of the system of reference means that the theory consists only of relations between physical objects. Metaphysical objects like space and time do not exist in such an interpretation. A theory like this is called relationalist. This idea will be implemented in subsection 3.1.3. Thus, in the first subtype of the substantivalist sense of absoluteness, point substantivalism, space-time in general relativity is relational.

[9]This is the set of diffeomorphisms with function composition, satisfying the group axioms. A diffeomorphism is the same as the above mentioned smooth transformation.

Subtype II: Geometry Substantivalism Even though point substantivalism is not a suggested interpretation, the Einstein field equations can still yield solutions of empty universes (for instance the De Sitter vacuum solutions, discussed in subsection 2.3.2). Vacuum solutions seem to tell us that space-time is absolute in the second substantivalist sense: space-time exists, in addition to physical objects. This means that the ontology of the world according to general relativity is physical objects *and* space-time (and the relations between these objects). From the existence of vacuum solutions to the Einstein equations, we can conclude that space-time can exist independently of matter. In that sense it is absolute. One might say space-time is a container of matter; this is substantivalism.

Even though it seems that we have reached a conclusion, the substantivalist interpretation of general relativity is nevertheless outdated or simply naive. In section 3.1.3 the 'correct approach' is discussed and the conclusion will be drawn, despite the above, that general relativity is not a substantivalist theory.

3.1.2 Hole argument

An important episode in the development of general relativity as well as in the philosophy of space and time is the hole argument, which was formulated in its original form by Einstein in 1913. From this argument Einstein concluded that the demand of general covariance is a bad idea. General covariance is the invariance of the form of physical laws under arbitrary differentiable coordinate transformations.[10] This means that if a metric is smoothly transformed, the resulting metric describes the same physical situation. Einstein thought this was a necessity for a fully relativised mechanics.

A modern formulation of the hole argument proceeds as follows:

Consider a space-time with some configuration of matter, $T_{\mu\nu}(x^\lambda)$, which defines some metric field $g_{\mu\nu}(x^\lambda)$. Define a 'hole' $h \subset M$ as a connected[11] finite part of the space-time manifold, where there is no matter ($T_{\mu\nu}(x^\lambda) = 0$ for all $x^\lambda \in h$). Now smoothly transform the coordinates in the hole $x^\lambda \mapsto x'^\lambda$, but leave the space-time coordinates outside the hole unchanged, such that the transition from inside the hole to outside the hole is smooth as well. The field values will be different, thus $g_{\mu\nu}(x^\lambda)$ will become $g'_{\mu\nu}(x'^\lambda)$. This is called a passive transformation. Now we may replace the x'^λ by x^λ, because it is only a *label.* We can always do this, but it is crucial to the argument. The resulting transformation is called an active transformation. Space-time points get moved around, because if x^λ are the coordinates of point $p \in M$, they are no longer the coordinates of p after the transformation, but of, say, $p' \in M$, provided this point is in the hole. Now $g_{\mu\nu}(x^\lambda)$ is transformed to $g'_{\mu\nu}(x^\lambda)$. This means that all matter (which is outside h) does not uniquely determine what will happen inside the hole, since,

[10]This is also referred to by 'smooth transformations' and 'diffeomorphisms'.

[11]Connectedness means that the described part of space-time cannot be partitioned into disjoint open sets.

apparently, $g_{\mu\nu}(x^\lambda)$ and $g'_{\mu\nu}(x^\lambda)$ have in general different fields-values. Thus, $T_{\mu\nu}$ does not uniquely determine $g_{\mu\nu}(x^\lambda)$.

The fact that one can perform an active coordinate transformation, without changing the form of the physical laws, is called *general covariance*. The indeterminate character of the solutions in the hole violate Mach's principle, thus led Einstein to believe that general covariance is a bad idea. Because, if there is no general covariance, one cannot formulate a hole argument and thus Mach's principle is not violated.

At least according to point substantivalists $g_{\mu\nu}(x^\lambda)$ and $g'_{\mu\nu}(x^\lambda)$ are physically different, because a point substantivalist believes in the existence of space-time points, independent of matter. These are ontologically different from each other. In the active interpretation of the diffeomorphism points are relocated, which means that for a substantivalist something is different and therefore for him diffeomorphic models are not equivalent.

However, the differences between diffeomorphic models are not empirically accessible, whether you are a substantivalist or a relationalist. In the latter case there is no ontological difference either. That is, a relationalist talks about coordinate independent metrics, called *geometries*.[12] In such a formulation only physical events (e.g. the interaction between an electron and a photon) are invariant under diffeomorphisms. The different coordinate dependent metrics, which have these events as invariants, are called equivalent. It depends on how strong one feels about this equivalence in order to make interpretative sense of a space-time theory. Roughly the following two positions can be taken.

The first position is to assume that all models, including diffeomorphic models, of space-time are different. It is the position of a *point substantivalist*, one who believes that space-time points have their own individual identities. For him diffeomorphic models are only empirically the same and differ ontologically. Considering the hole argument this means that the metric in the hole is indeterminate. There are different solutions to the metric of the hole, even though one would hope that the matter (which is outside the hole) determines the metric inside the hole. The fact that there are different solutions is a form of *indeterminism*.

The second option is to accept *Leibniz equivalence*. It is the position of a *relationalist*. Leibniz equivalence is the assumption that observationally indistinguishable models represent the same physical situation. The metrics in (and outside) the hole are observationally indistinguishable. Thus, if Leibniz equivalence is accepted, the metrics represent the same physical situation. They are ontologically the same. As a result there is no indeterminism in the above mentioned sense.

[12]In the context of the hole argument we will talk about the *metric* or a *model* when we mean the coordinate dependent metric. The equivalence class of models will be called the *geometry*. With the word *equivalence* (without any adjective) Leibniz equivalence is meant. In the discussed paper of Earman and Norton, Leibniz equivalence is defined as *"Diffeomorphic models represent the same physical situation"* [EN87, p. 522].

3.1.3 Reinterpretation of general relativity

It seems that what we see around us is space and we experience a flow of time. What we actually see is objects moving through this space. A modern, natural way to interpret general relativity is to assume that there is no space or time at all. The only physical objects are the metric field and the particles.[13] It seems very natural to say that space and time exist independently from matter, but this is just an illusion. Interactions between fields and particles lead us to believe that there is space and time. However, it is not actually there, according to the interpretation of general relativity that is here discussed: there are only the field, particles and relations.[14] An example is that of the crossing of a particle with a field. Intuitively this would be the path or world line of the particle, not so much through space-time, but more like a continuous set of coincidences with the field. This will be treated in the next subsubsection.

The physicist Carlo Rovelli has the point of view that there is no space-time but only the field. To understand this, we must not be substantivalists. For this we should not start with a manifold, but we should build up the theory the other way around. That is, we should start with what is actually there: the field and the particles. The whole construction with the manifold is just to arrive in a mathematically insightful way to the correct formalism. In the end there is nothing real about the manifold, there is just the field and particles, Rovelli argues. This means that particles or the field do not have coordinates in space-time, but there are only relations between them. The following example illustrates this idea.

Consider a model of two particles moving through space-time, coinciding once at a certain point in space-time. One might perform arbitrary smooth[15] transformations on the coordinates and consider these transformations as active, like in the hole argument. This means that, in a substantivalist interpretation of space-time, space-time points get moved around. These smooth transformations are *diffeomorphisms* and thus the different models are referred to as *diffeomorphic models*. All diffeomorphic models are empirically equivalent to this model, as long as there is still one coincidence (at any 'location').

If Leibniz equivalence is assumed, these models are ontologically equivalent

[13] Cf. e.g. [Rov04, pp. 71–75].

[14] One may go further and say that there are only relations. This is ontic structural realism. We will not assume this as a valid thesis in this paper, because of the problem of the existence of relations without relata, as well as the fact that the treatment of structural realism is outside the scope of this paper.

[15] A *smooth manifold* is a topology with a differential structure, which has derivatives of all orders. Such a manifold is said to be C^∞. A *smooth transformation* is also C^∞ and leaves the smoothness of the manifold intact. They are called *diffeomorphisms*. A Riemannian manifold is a manifold which does not have to be smooth everywhere on its domain (for instance, black holes exist). Smoothness is (and can) only (be) invariant under diffeomorphisms everywhere where the manifold is smooth. For simplicity, only smooth manifolds are considered.

as well. Therefore there is only one actual model that describes a coincidence of two particles, resulting in a simple structure. This actual model is an equivalence class of diffeomorphic models. Add a metric field. (The significance for this will become clear in the next subsubsection.) Now the particles do not only have relations with respect to each other, but also to the metric field. The resulting structure is much more interesting. There is a continuum of coincidences between the particles and the metric field. Paths of the particles with respect to the metric field emerge. The resulting structure is richer than before. If we now add more particles and fields, the structure gets even richer. From this experiences emerge that lead us to believe that space and time exist.

Consider the following analogy to the two-particles example. Suppose you have three men $\{A, B, C\}$ and three women $\{D, E, F\}$. How many distinct heterosexual marriages are possible? Nine, one would say:

$$\begin{array}{ccc} \{A,D\} & \{A,E\} & \{A,F\} \\ \{B,D\} & \{B,E\} & \{B,F\} \\ \{C,D\} & \{C,E\} & \{C,F\} \end{array}$$

This is the usual way to describe these relations. A substantivalist of some abstract "marriage space" on the other hand would assume slots exist $(\cdot, \cdot)$ which must be filled by people. This would yield eighteen different marriages:

$$\begin{array}{cccccc} (A,D) & (A,E) & (A,F) & (D,A) & (E,A) & (F,A) \\ (B,D) & (B,E) & (B,F) & (D,B) & (E,B) & (F,B) \\ (C,D) & (C,E) & (C,F) & (D,C) & (E,C) & (F,C) \end{array}$$

The substantivalist would say that (A, D) is ontologically different from (D, A). However, he would agree with the relationalist that they are empirically equivalent. Obviously the ontological difference does not make much sense. It is just an artifact of our representation of 'marriage', just like the description of particles on space-time points is an artifact of our representation.[16]

Extended point-coincidence argument

Analysing Einstein's point-coincidence argument (as discussed in section 2.2.1), initially it seems that it resolves the hole argument in a satisfying way. Under smooth transformations coincidences are preserved, and these are the only things that might have an empirical effect.[17] Therefore general covariance is not problematic anymore and thus the hole argument loses its grip. However, it is not

[16]To make it more explicit, the analogy between this (marriage example) with the previous paragraph (particles example) is this: the space-time in the particles example is the marriage space in the marriage example. The two particles in the particles example is the six people in the marriage example. Similarly to the marriage example in which there are only relations between people, there are only relations between particles in the particle example (and in extension to that also with respect to the metric and the hereafter introduced particles).

[17]The opposite is not true: there are coincidences which have no observable effects.

immediately clear whether precisely this preservation of coincidences is what is needed. If only such point coincidences are to be preserved, the resulting allowed transformations might be much wider than smooth coordinate transformations. The point-coincidence argument is certainly enough to allow diffeomorphisms, but doubt might be raised whether the argument is too liberal, that is, it might allow more transformations than the ones essential to general covariance. Let us analyse the set of point-coincidences conserving transformations on the one hand and the set of smooth transformations on the other hand.

The set of transformations which only conserve point coincidences between particles is very large.[18] Every coincidence might be relocated anywhere in space-time. There are no restrictions. Even when two particles coincide not just on one space-time point, but for a short duration (say of proper time $\delta\tau$), each point of this coinciding world line can be relocated anywhere in space-time. A consequence is that continuity of the world line is lost. The set of point coincidences is simply the largest possible set of transformations on a space-time. This set of transformations will now be compared with the set of diffeomorphisms.

The set which *smoothly* transforms the metric field leaves the continuity (and differentiability of all orders) of world lines intact, but is just like the set of permutations very large.[19] Of course, when arbitrary diffeomorphisms (smooth transformations) are allowed, the location of world lines of particles are not uniquely determined, but diffeomorphisms cannot change the smoothness of these lines.

The latter set of transformations is much smaller than the set which only keeps intact coincidences of particles, because generally an arbitrary point transformation breaks world lines, while diffeomorphisms do not. See appendix B for a mathematical foundation. This illustrates that Einstein's point-coincidence argument is more liberal than necessary, i.e. the conservation of point-coincidences leads to an even larger set of transformations than is needed to justify general covariance. For Einstein it was enough to mention that at least some space-time points are invariant under smooth transformations. He did not intend to give a full characterisation of what is and what is not invariant. However, if we want to give a full characterisation of what is invariant, a smoothness restriction is necessary to make the point-coincidence argument compatible with general co-

[18] The set of coincidence preserving transformations is the set of permutations of all points on the manifold M. A *permutation* is an arbitrary number of transpositions of space-time points. Given a certain model of space-time, there is no restriction on permutations of points. This gives a set of cardinality $\aleph_2$. The number of points in the manifold M is $|M| = \aleph_1$, the same cardinality as of real numbers. Here follows a proof sketch. The set of permutations is of size $|M|!$, which is larger than $2^{|M|}$ but smaller than $|M|^{|M|}$. The lower and upper bounds are both $\aleph_2$ (generalised continuum hypothesis is assumed), thus this is the size of the set of permutations of all points in M as well. See appendix B for further details.

[19] Every point on the manifold might be moved to any other point under diffeomorphisms, thus for every point there is a possible transformation which is $|\mathbb{R}| = \aleph_1$ of cardinality. This might be done for any point. All combinations of transformations gives a size of the set of transformations of the order of ${\aleph_1}^{\aleph_1} = \aleph_2$. See appendix B for a proof strategy.

variance.

Still, it is not enough, because the admissible transformations are performed on the four-metric (a four-dimensional object, instead of the one-dimensional world lines or zero-dimensional instantaneous coincidences). The whole metric must be transformed smoothly. Let us see how this can be achieved if we treat the metric as one of the coincidable objects.

If the metric is included in the objects of the point-coincidence argument, making this argument compatible with general covariance gets easier. Imagine a metric and one particle. These objects coincide throughout the life span of the particle (say its proper time $\Delta\tau$). This forms a smooth world line (like the endured coinciding of the two particles). Again we must introduce a smoothness restriction which keeps the world line of the particle, with respect to the metric, intact. We might introduce more particles which coincide with the field and possibly with each other, and we might even introduce extra fields. If now arbitrary transformations are made in which all coincidences are invariant, the resulting set of transformations is significantly restricted and Einstein's original point-coincidence argument is extended (or restricted, if you prefer to refer to the transformation restriction, rather than the extension of the set of coincidable objects) in such a way that it is perfectly compatible with general covariance.[20]

The smooth transformations on the hole in the hole argument precisely leave the coincidences in the extended point-coincidence argument unchanged. The extended point-coincidence argument is equivalent to accepting Leibniz equivalence. In this way the problem of the hole argument is trivially solved, in accord to Einstein's intentions.

Conclusion

On the basis of Rovelli's interpretation, looking back to Friedman's versions of 'absolute', we can conclude the following. As was shown, space-time of the general theory of relativity is not absolute in any sense. It is relative, dynamical and it is relational. It is relational, because it is not substantivalist in either sense: manifold points do not have an individual identity and the solutions to the Einstein equations are just solutions of the coordinateless metric field or the geometry, not of an independently existing space-time container.

[20] Alternatively, this argument might be made by first introducing the metric field and only later thinking about smoothness. Again, imagine a metric and one particle. Even though all coincidences now form a set which is connected and not countable, this does not imply that the set of admissible transformations in this case is smaller than the set of diffeomorphisms (as discussed in the next paragraph). Only if we demand the smoothness of world lines through the field, the set of admissible transformations will be significantly smaller. If we demand the smoothness of the metric field (which we will assume to be locally homeomorphic to $\mathbb{R}^4$), our point-coincidence argument is extended in such a way that the set of admissible transformations is the same as the one in general relativity (the set of diffeomorphisms).

If Friedman's list is a complete list of relevant definitions of 'absolute', this would mean that in general relativity there is no absolute space-time in any sense. However, Friedman's distinction is insightful and the outcome of this analysis was not trivial.

3.2 Unobservables

It has been shown in the previous section, that in general relativity space and time are idle metaphysical concepts, at least in the sense of space-time points. One might be tempted to say that therefore general relativity is a Machian theory. There is one problem with this point of view. We do not only want a Machian theory to speak of no 'absolute' space-time (of any kind), but it should not contain *any* 'theoretical' or 'unobservable' objects as well. The geometry or metric in general relativity is such a theoretical object. On the basis of this argument we will conclude that the general theory of relativity is not a Machian theory.

3.2.1 Point coincidences and observables

Einstein introduced the point-coincidence argument in correspondence with Paul Ehrenfest and Michele Besso. The first article in which he set forth the argument was in his "The Foundation of General Relativity".[21] He wrote:

> "All our statements about space and time always come down to the determination of space-time coincidences. If for instance only the motion of matter points would take place, these would eventually not be observable, unlike the meeting of two or more of these points." [22]

In the first sentence Einstein does not talk about observables yet. He says that the space-time coincidences are the only things that matter for statements about, or a description of, space and time. The specific description he talked about is the general theory of relativity. Thus, what he says is that general relativity is about space-time coincidences and nothing else.

In the second sentence Einstein uses the word *observable* ('beobachtbar'). He says that the meeting of two or more matter points, the space-time coincidences, are observables.

If general relativity is about space-time coincidences[23] and all space-time coincidences are observable, we must conclude that general relativity is a theory

[21] "Die Grundlage der algemeinen Relativitätstheorie" [SKK96, pp. 283–339]

[22] "Alle unsere zeiträumlichen Konstatierungen laufen stets auf die Bestimmung zeiträumlicher Koinzidenzen hinaus. Bestände beispielsweise das Geschehen nur in der Bewegung materieller Punkte, so wäre letzten Endes nichts beobachtbar als die Begegnungen zweiter oder mehrerer dieser Punkte."

[23] In a precise way, iff space-time coincidences in the extended sense are meant.

about observables. This brings us to believe that general relativity is a theory of only experience. This is Norton's second demand for a theory to be Machian. We already showed that absolute space, time and motion are metaphysical idle notions in general relativity. That is Norton's first demand. This means that both demands are fulfilled and we can conclude that the general theory of relativity is a Machian theory. However, there might be a flaw in this argument.

3.2.2 The term 'observable'

Einstein equates 'observable' with 'point coincidence'. It is not clear whether this is a valid equivalence. Einstein's idea of observability may be incompatible with observable in a Machian sense. Let us distinguish two different types of observability:

- Observability in the objective sense (physically real).
- Observability in the Machian sense.

Observability in the objective sense is the way Einstein uses the word observable. It is something which is described by the theory under consideration. Maybe the theory even *defines* observability. A priori it is not clear whether the term 'observable' in different theories are either intuitively or logically compatible with each other. For instance, in Newtonian space-time all space-time points are objective observables, while in general relativity a lot of these previously observables can be transformed away because of diffeomorphism invariance.

However, a Machianist does not use 'observability' in the above sense. He would use 'observability' in the sense of actually observable for some observer. Most coincidences, as used by Einstein, are not observable. This is slightly ironic, because Einstein was led partly by Mach's ideas to the general theory of relativity.

Observability in the Machian sense is common sense observability. A chair is an observable object, for the chair reflects light, which may fall on our retina and sends a signal to our brain, making us conscious of the existence of the chair. In this process a lot of space-time coincidences take place, which makes the chair an objective observable as well. But now consider an interaction between two electrons. Only in the case it happens to radiate light, with a sufficiently large intensity and in the visual spectrum, which falls in our eye, it might be called 'observable', but in almost all other cases of particles meeting, it is not observable in a Machian sense.

Einstein uses observability in the objective sense, which is much wider than the Machian sense. Thus, the difference between the general theory of relativity and a typically Machian theory[24] is that there still are objects which are not

[24]This is the full-fledged theory which Mach would have constructed according to his own positivistic principles.

observable in the Machian sense in general relativity. Therefore, general relativity does not cut it as a truly Machian theory.

3.2.3 Physical objectivity

In the context of general relativity we must talk about space-time coincidences, not about observables. We have seen an interpretation of general relativity, where the point-coincidence argument was extended in such a way that it is closer to general covariance. This is the interpretation of Rovelli as described in subsection 3.1.3.

There is another way to extend Einstein's point-coincidences to make the set of coincidences just as large as Rovelli's, that is, a four-manifold. For this we do not consider the metric field (the geometry) as a coincidable object; only particles will be considered as such. To probe the metric field in a part of space-time we can use test particles. These test particles should not significantly change the metric. We can shoot as many test particles as we want, and maybe let them interact with each other. This gives a complete description of the metric.

If we do not use these test particles, the metric (geometry) will just be as real and uniquely determined as when we do. This is physical objectivity. In classical theories it does not matter whether you perform a measurement or not, as long you do not adjust the model as a result of the test particles. Therefore, we might talk about an arbitrary large collection of potential or modal models of test particles through any metric. This gives a set of coincidences just as large as Rovelli's set, without the introduction of the metric field. The only problem with this alternative is that you have to become a realist about potential models. This alternative approach does not bring us any closer to the experience demand of John Norton and thus not to Machianism.

Concluding remarks

The general theory of relativity might be interpreted as partly Machian, because space, time and motion can be seen as metaphysical idle notions. Specifically, the geometry in general relativity is relational, relative and dynamical, thus not absolute in any sense. However, it is not fully Machian, because it is not a pure description of experience. Even when we only consider Einstein's original argument, most point-coincidences are not observable in a Machian sense. When we extend these coincidences to match the actual physical objects in general relativity, we end up in an even worse situation: we must either interpret the metric field as an object which has significance in observability or we must be a realist of possible but not actual models of the universe. Therefore, general relativity is only partly a Machian theory.

Conclusion

In this thesis Mach's philosophy of mechanics and relationalism have been analysed, with the goal of answering the following questions. How do Mach and the ideas of modern relationalists relate to one another? Is the general theory of relativity a relational theory? Is it Machian? This debate has been discussed in a historical way, in which written ideas of scientists and philosophers were analysed. It has also been discussed in a technical, foundational way, in which theories were studied. Neither of these approaches have given conclusive answers to my research questions.

The historical enquiry in Mach has given insight into Mach's ideas about space, time and mechanics and other's interpretations thereof. It is difficult to say what Mach's intentions were. On the one hand, part of Mach's work shows that he tried to formulate a new theory. On the other hand, the opposite, that he merely tried to reformulate Newtonian mechanics, can also be maintained when looking at Mach's work as a whole. From the fact that he wrote down a beginning of a new theory, does not follow that he actually intended to formulate a new theory. He might have been happy with a rewritten relational version of Newtonian mechanics. However, it is clear that he wished for a fully relational theory in which space and time do not exist independently from other objects.

Einstein was motivated by several of Mach's ideas. From this one cannot conclude that general relativity is relational, because the full formulation of the theory does not have obviously Machian properties. However, from the study of general relativity and relational interpretations (specifically Rovelli's) it can be concluded that general relativity can, in a natural way, be interpreted as a relational theory. An interpretation cannot follow from a theory, thus relationalism is not a 'mandatory interpretation' of general relativity. It is at most *suggested* by the formalism.

From this does not follow that general relativity is Machian. The term 'Machian' is defined as being relational, that is, the idea that space and time are metaphysical idle notions, as well as the desire to provide economical descriptions of experience. The theory contains unobservables like the metric field. Such an object is not fully accessible to our senses and therefore the general theory of relativity is not a pure description of experience. This means that general relativity is partly Machian at most.

The historical as well as the foundational enquiry into the question of the relation between the philosophies of Mach and the interpretation of general relativity is sensitive to the interpretation of Mach's intentions as well as interpretation of the theory. Therefore it is difficult to draw an objective conclusion. Furthermore, Mach and the modern relationalist are not easily compared, because the modern relationalist possesses the knowledge of general relativity and its interpretations. Mach did not have such a full-fledged theory which can be interpreted in a relationalist way. Therefore it is not a fair comparison. However, roughly Mach and the modern relationalist talk about the same things. Relationalism for Mach is that space and time are not independent objects, like Newton presupposed. The modern relationalist strives to the same thing.

Appendix A

Classical space-times

Earman has distinguished different classical space-times [Ear89, pp. 27-40]. He starts from a space-time with almost no structure, Machian space-time. In this space-time, space only has a Euclidean metric and there is absolute simultaneity. Note that a Euclidean metric structure does not imply absolute structures like inertial frames or even the existence of points. On this space-time with almost no structure Earman builds more structure, working additively to full Newtonian space-time, where points exist and have an identity over time. This means that for a given $t \in \mathbb{R}$, any space-time point $(x, y, z)(t)$ is the same as $(x, y, z)(t')$, for any $t' \in \mathbb{R}$. Here follows an overview of Earman's different space-times.

Machian space-time is a four-manifold, partitioned by three-dimensional hypersurfaces that are topologically $\mathbb{R}^3$, such that two events are simultaneous if and only if these lie in the same hypersurface. Machian space-time does not possess a time metric, thus there is no notion of 'velocity'.

Leibnizian space-time is Machian space-time with a time metric. Because of the time metric, relative speed between two particles can be defined.

Maxwellian space-time is Leibnizian space-time with a standard of rotation. This means that the time dependence of rotation is killed, giving space the structure of absolute rotation.

Galilean (or neo-Newtonian) space-time is Maxwelian space-time with a standard of translation. This gives space the structure of spatial acceleration.

Full Newtonian space-time is Galilean space-time with absolute space. It might be represented as a foliation of the product of $\mathbb{E}^3$ (Euclidean space) with $\mathbb{R}$ (time).

At several places in this thesis these definitions are used, but only when it is useful to make these distinctions. In most cases the distinction between 'relational space-time' (Machian or Leibnizian), Galilean space-time and full Newtonian space-time is enough.

Appendix B

Size of sets of transformations

This appendix is a footnote to a naive reading of Einstein's point-coincidence argument. The goal is to find out, given a differentiable manifold M, how big the set of diffeomorphisms is and how big the set of permutations of all points of M is.

This proof strategy is given within Zermelo-Fraenkel set theory with the axiom of choice (ZFC). The generalised continuum hypothesis is assumed. This hypothesis is not provable or refutable within ZFC, but it is assumed in this appendix, because this is common in ZFC and it gives a simple way to talk about the cardinality of a set.

Point transformation

Every point on an n-manifold might be moved to any other point under diffeomorphisms, thus for every point there is a possible transformation which is $|\mathbb{R}^n| = \aleph_1$ of cardinality. This might be done for any point. All combinations of transformations gives a size of the set of transformations of the order of ${\aleph_1}^{\aleph_1} = \aleph_2$, which is just as big as the cardinality of the set of permutations. We must bear in mind that there are by definition C^∞ restrictions for diffeomorphisms, but these restrictions do not have an influence on the cardinality of the set. How can we see this? See what happens with finite sets if we either permute or transform the set smoothly. Transforming a finite set without differentiable structure smoothly does not make any sense, but let us try to give the finite sets a structure which resembles (for simplicity sake) a one-dimensional manifold.

Finite set

Consider the one-manifold $]0,1[\subset \mathbb{R}$ (an open line segment). A diffeomorphism on this manifold is an arbitrary deformation of the line segment, i.e., all contractions, expansions and translations of the points are allowed, as long as its neighbouring relations are preserved. (Think of it as a rubber band, which does not snap.) In one dimension this is the preservation of order. The difficulty with finite sets, however, is that for such a trivial structure, there is nothing to deform.

Consider the set $\{1, 2, 3, 4\}$. A trivially ordered version of this is $(1, 2, 3, 4)$. There are no transformations which preserve its ordering, so our set of 'diffeomorphisms' would be empty. In a real manifold (like our $]0, 1[$) something funny is going on. The object is flexible, because the points can move with the preservation of its order. If we want to mimic this in our finite set, we must put some 'holes' in our ordered set. 'Holes' are points without a label, thus for instance, $(1, 2, \cdot, 3, \cdot, \cdot, 4)$ is an ordered set with three holes. Now we can transform this object to, say $(1, 2, 3, \cdot, \cdot, \cdot, 4)$. We have moved 3 one unit to the left, contracting the space between 2 and 3 and expanding it between 3 and 4. This is the relevant analogy for a diffeomorphism: transformations which preserve order. In this specific case we have $\binom{7}{4} = 35$ possible transformations. The general formula for the number of allowed order preserving transformations, including the identity, is $|\mathrm{Diff}| = \binom{N}{m}$, where N is the number of allowed 'locations' and m is the number of labels to be moved around.

On the other hand, when allowing arbitrary permutations as well (not preserving the order), we get $4! \cdot 35 = 840$. Generally, the number of permutations is $|\mathrm{Perm}| = m! \cdot \binom{N}{m}$. In this example there are many more permutations than 'diffeomorphisms'.

If larger sets are considered, this will show that the number of 'diffeomorphisms' grows rapidly, but the number of permutations grows even disproportionally faster than the number of 'diffeomorphisms'.

One-manifold

If we now move on to the one-manifold, the calculation is not that trivial. We know that $\mathbb{Q}$ lies dense in $\mathbb{R}$. It would be interesting to look how rational number can be shifted on our manifold $]0, 1[$. If we now generalise our formulae for the size of $\mathbb{Q}$ and $\mathbb{R}$, the size of diffeomorphisms on the one-manifold is:

$$|\mathbf{Diff}| = \binom{\aleph_1}{\aleph_0} = \aleph_1.$$

(This might be shown with the Stirling approximation, or a method similar to the estimation in footnote 18 of chapter 3.) The size of the set of permutations is:

$$|\mathbf{Perm}| = \aleph_0 \cdot \binom{\aleph_1}{\aleph_0} = \aleph_1.$$

The factor $\aleph_0$ is not relevant for the cardinality, but that fact is not relevant for Einstein's argument. The point is that the two sets differ and that Einstein's point coincidence conserving group contains many transformations which are not contained by the group of diffeomorphisms.[1]

[1]Above formulae might be applied to arbitrary four-manifolds, specifically Riemannian manifolds. Of course all this is just a strategy to derive our conclusion. There are several non-trivial steps to make it a proof (sketch). One of these steps is the generalisation from finite to infinite sets.

Acknowledgements

I would like to thank several people who helped me with my research and the resulting thesis. First of all I want to thank Jeroen van Dongen and Dennis Dieks who supervised my research. They were very helpful by giving constructive critique to my writings.

Furthermore, I thank Carlo Rovelli, Julian Barbour and John Norton for their insightful comments that I received by e-mail. I also received useful comments from several of my good friends, Marijn Schouten, Yves van Gennip, Wilfred de Bondt, Fedde Benedictus and Bernhard Pos.

Last but not least I thank Jasmijn van Huis. Without her moral support and extensive help with my English thus making my thesis readable, I could not have written this more than satisfactory thesis.

Bibliography

[BP95] Julian Barbour and Herbert Pfister, editors. *Mach's Principle: From Newton's Bucket to Quantum Gravity (Einstein Studies)*, volume 6 of *Einstein Studies*, Boston, 1995. Birkhäuser, Boston University.

[Car97] Sean M. Carroll. *Lecture Notes on General Relativity.* University of California, Santa Barbara, Dec 1997.

[DiS90] Robert DiSalle. Book review: Mach I, Mach II, Einstein, und die Relativitatstheorie. Eine Falschung und ihre Folgen Gereon Wolters. *Philosophy of Science*, 57(4):712–723, 1990.

[Ear89] John Earman. *World Enough and Space-Time: Absolute versus Relational Theories of Space and Time.* The MIT Press, Cambridge, Massachusetts, 1989.

[EN87] John Earman and John Norton. What price spacetime substantivalism? The hole story. *The British Journal for the Philosophy of Science*, 38(4):515–525, 1987.

[Fri83] Michael Friedman. *Foundations of Space-Time Theories.* Princeton University Press, 1983.

[Hol73] Gerald Holton. *Thematic Origins of Scientific Thought.* Harvard University Press, Cambridge, Massachusetts, 1973.

[Jan05] Michel Janssen. Of pots and holes: Einstein's bumpy road to general relativity. *Ann. Phys. (Leipzig)*, 14, Supplement:58–85, 2005.

[KSKR95] Martin J. Klein, R. Schulmann, A.J. Kox, and Jürgen Renn, editors. *The Swiss Years: Writings, 1912–1914*, volume 4 of *The Collected Papers of Albert Einstein.* Princeton University Press, Princeton, New Jersey, 1995.

[Mac21] Ernst Mach. Die prinzipien der physikalischen optik. *Isis*, 4(3):560–562, 1921.

[Mac60] Ernst Mach. *The Science of Mechanics.* The Open Court Publishing Company, LaSalle, Illinois, 1960.

[New87] Isaac Newton. *Philosophiæ naturalis principia mathematica.* J. Societatis Regiae ac Typis J. Streater, London, 1st edition, 1687.

[Pai83] Abraham Pais. *Subtle is the Lord: The Science and the Life of Albert Einstein.* Oxford University Press, Oxford, November 1983.

[Ray87] Christopher Ray. *The Evolution of Relativity.* J W Arrowsmith Ltd, Bristol, 1987.

[Rov04] Carlo Rovelli. *Quantum Gravity.* Cambridge University Press, Cambridge, November 2004.

[SJI+02] R. Schulmann, Michel Janssen, József Illy, Christoph Lehner, and Diana Kormos Buchwald, editors. *The Berlin Years / Die Berliner Jahre: Correspondence / Briefwechsel, 1918–1921*, volume 7 of *The Collected Papers of Albert Einstein.* Princeton University Press, Princeton, New Jersey, 2002.

[SKJI98a] R. Schulmann, A.J. Kox, Michel Janssen, and József Illy, editors. *The Berlin Years / Die Berliner Jahre: Correspondence / Briefwechsel, 1914–1917*, volume 8a of *The Collected Papers of Albert Einstein.* Princeton University Press, Princeton, New Jersey, 1998.

[SKJI98b] R. Schulmann, A.J. Kox, Michel Janssen, and József Illy, editors. *The Berlin Years / Die Berliner Jahre: Correspondence / Briefwechsel, 1914–1917*, volume 8b of *The Collected Papers of Albert Einstein.* Princeton University Press, Princeton, New Jersey, 1998.

[SKK96] R. Schulmann, A.J. Kox, and J. Klein, editors. *The Berlin Years: Writings, 1914–1917*, volume 6 of *The Collected Papers of Albert Einstein.* Princeton University Press, Princeton, New Jersey, 1996.

[Wal84] Robert M. Wald. *General Relativity.* University Of Chicago Press, Chicago, June 1984.

[Wol87] Gereon Wolters. *Mach I, Mach 2, Einstein Und Die Relativitatstheorum.* Walter de Gruyter & Co, New York, 1987.